Global Existence of Solutions to Coupled $\mathcal{PT}$-Symmetric Nonlinear Schrödinger Equations

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Abstract We study a system of two coupled nonlinear Schrödinger equations, where one equation includes gain and the other one includes losses. Strengths of the gain and the loss are equal, i.e., the resulting system is parity-time ($\mathcal{PT}$) symmetric. The model includes both linear and nonlinear couplings, such that when all nonlinear coefficients are equal, the system represents the $\mathcal{PT}$-generalization of the Manakov model. In the one-dimensional case, we prove the existence of a global solution to the Cauchy problem in energy space H^1, such that the H^1-norm of the global solution may grow in time. In the Manakov case, we show analytically that the L^2-norm of the global solution is bounded for all times and numerically that the H^1-norm is also bounded. In the two-dimensional case, we obtain a constraint on the L^2-norm of the initial data that ensures the existence of a global solution in the energy space H^1.

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1 Introduction

A system of two coupled nonlinear Schrödinger (NLS) equations represents the basic model of propagation of weakly dispersive waves having vectorial nature, i.e., characterized by two components, below designated by u and v. This model is relevant for almost all applications of the nonlinear physics. When the system is supplied by balanced gain and losses, as well as by the linear coupling, characterized respectively by positive coefficients $\gamma > 0$ and $\kappa > 0$, it reads

$$iu_t = -u_{xx} + \kappa v + i\gamma u - (g_{11}|u|^2 + g_{12}|v|^2)u,$$
$$iv_t = -v_{xx} + \kappa u - i\gamma v - (g_{12}|u|^2 + g_{22}|v|^2)v, \tag{1}$$

where all the nonlinear coefficients, i.e., g_{11}, g_{22} and g_{12} are real. Model (1) describes wave guiding in structures obeying parity-time ($\mathcal{PT}$) symmetry [1]. Here the parity symmetry is defined by the mapping $\mathcal{P}(u, v) = (v, u)$. Time reversal operator $\mathcal{T}$ is defined by the map

$$\mathcal{T}(u(t), v(t)) = (\bar{u}(-t), \bar{v}(-t)).$$

This means that if $g_{11} = g_{22}$, then the nonlinearity is also $\mathcal{PT}$ symmetric, i.e., for any solution $(u(t), v(t))$ defined in the symmetric interval $[-t_0, t_0]$, there also exists another solution

$$u_{\mathcal{PT}}(t) = \bar{v}(-t) \quad \text{and} \quad v_{\mathcal{PT}}(t) = \bar{u}(-t).$$

Hereafter an overbar stands for the complex conjugation. Notice that the nonlinear coupling coefficients g_{12} in (1) are considered to be equal in the both equations to ensure that the system is Hamiltonian in the absence of gain and loss with $\gamma = 0$.

Coupled NLS equations (1) with $\mathcal{PT}$-symmetric nonlinearity $g_{11} = g_{22}$ and with $g_{12} = 0$, introduced independently in [2] for the constant gain-and-loss coefficient and in [3] for gain and losses localized along the waveguide, can be viewed as an extension of the nonlinear dimer model [4, 5]. In the subsequent works [6, 7] more general nonlinearities, including ones with $g_{12} \neq 0$, were thoroughly studied. It was found that the model supports bright solitons [2, 7], dark solitons [6], Peregrin solitons and rogue waves [7], and breathers [8]. For all refereed cases it is important that the nonlinearity obeys $\mathcal{PT}$ symmetry (for the relevance of $\mathcal{PT}$ symmetry of the nonlinearity, see [9]).

System (1) is integrable if $\kappa = \gamma = 0$ and $g_{11} = g_{22} = g_{12}$, representing the well-known Manakov model [10] (see also [11] for review of these integrable systems). It turns out, that this particular type of (all equal) nonlinearities is peculiar also for the $\mathcal{PT}$-symmetric model which accounts for gain and losses, as well as for the linear coupling. More generally, the role of the type of nonlinearity (i.e., non $\mathcal{PT}$-symmetric, $\mathcal{PT}$-symmetric, and Manakov type) can be crucial for the long time behavior of the solution. Indication on such possibility stems form the recent results on the stability of trajectories of the nonlinear $\mathcal{PT}$-symmetric dimer. Namely, the nonlinear dimer

$$iu_t = \kappa v + i\gamma u - |u|^2 u,$$
$$iv_t = \kappa u - i\gamma v - |v|^2 v, \tag{2}$$

has trajectories escaping to infinity (as $t \to \infty$) [12–14], while the $\mathcal{PT}$-symmetric Manakov dimer, i.e., the system

$$iu_t = \kappa v + i\gamma u - \left(|u|^2 + |v|^2\right)u,$$
$$iv_t = \kappa u - i\gamma v - \left(|u|^2 + |v|^2\right)v, \tag{3}$$

has all trajectories bounded [15], provided that $\gamma < \kappa$. Note that the later constraint is related to linear stability of the zero equilibrium in systems (1), (2), and (3). In the theory of $\mathcal{PT}$-symmetric systems, it is referred to as a condition of unbroken $\mathcal{PT}$ symmetry [1].

In the present paper, we address the existence and boundedness of solutions to the Cauchy problem for the generalized Manakov system (1) with different types of nonlinear coefficients. For the situations where analytical results are not available, we perform numerical simulations. We also extend the global existence results to the case of two spatial dimensions.

The paper is organized as follows. Main results in one dimension are presented in Section 2. They consist of two theorems, the first one about global well-posedness of the general model (1) and the second one is about bounds on the L^2-norm of the solution in the $\mathcal{PT}$-symmetric Manakov case. The two theorems are then proved in Sections 3 and 4. Section 5 reports numerical results illustrating that the H^1 norm is also bounded in the $\mathcal{PT}$-symmetric Manakov case, but it may grow to infinity in the $\mathcal{PT}$-symmetric non-Manakov case. Finally, extensions of these results to the $\mathcal{PT}$-symmetric Manakov system in two spatial dimensions are considered in the concluding Section 6.

2 Main Results

Because system (1) is semi-linear, it can be rewritten in an integral form by using Duhamel's principle. Applying the well-known method based on the contraction mapping principle (see [16] for details), one can immediately establish existence and uniqueness of local solutions of the Cauchy problem for system (1) in Sobolev spaces $H^s(\mathbb{R})$ for any $s > \frac{1}{2}$. In what follows, we are interested in the behavior of solutions in the energy space $H^1(\mathbb{R})$, which corresponds to $s = 1$ and appears to be most relevant for physical applications.

Our first result establishes existence of global solutions in the energy space for arbitrary choice of coefficients in the generalized Manakov system (1).

Theorem 1 *For any $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, there exists a unique global solution $(u(t), v(t)) \in C\left(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R})\right)$ of the generalized Manakov system (1) such that $(u(0), v(0)) = (u_0, v_0)$.*

We remark, that explicit examples of global solutions, including bright solitons, can be readily obtained in the case $g_{11} = g_{22}$ and $\gamma < \kappa$, since the substitution [2, 7, 17] $v = e^{i\delta}u$ with δ defined by $\sin(\delta) = -\gamma/\kappa$, reduces system (1) to the integrable nonlinear Schrödinger equation

$$iu_t = -u_{xx} + \kappa \cos(\delta)u - (g_{11} + g_{12}) |u|^2 u. \tag{4}$$

The H^1-norm of a global solution to the generalized Manakov system (1) remains finite but may grow as $t \to \infty$. This growth happens typically in the systems with gains and losses, at least for sufficiently large initial data, see e.g. [12–14]. It is then quite surprising that the Manakov system (1) with $g_{11} = g_{12} = g_{22}$ has bounded solutions in the L^2-norm if $\gamma < \kappa$ (i.e. under the condition of the unbroken $\mathcal{PT}$ symmetry). The next theorem formulates the corresponding result.

Theorem 2 *Assume $g_{11} = g_{22} = g_{12}$ and $\gamma < \kappa$. For any global solution $(u(t), v(t)) \in C(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ of Theorem 1, there exists a constant $Q_{max} > 0$ such that*

$$\sup_{t \in \mathbb{R}} \left(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 \right) \leq Q_{max}. \tag{5}$$

For $\gamma \geq \kappa$, there exists a global solution of Theorem 1 such that

$$\lim_{t \to \infty} \left(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 \right) = \infty. \tag{6}$$

Note that Theorem 2 generalizes the result of our previous work [15] devoted to the integrable $\mathcal{PT}$-symmetric Manakov dimer (3) that corresponds to the x-independent solutions of system (1) with $g_{11} = g_{22} = g_{12}$. In comparison, the result of Theorem 1 is novel and the proof is based on deriving apriori energy estimates for the H^1-norm of the local solutions and applying Gronwall's inequality.

Also note that the results of Theorems 1 and 2 do not clarify if the H^1-norm of the solution is also globally bounded in the case $g_{11} = g_{22} = g_{12}$ and $\gamma < \kappa$. Numerical simulations reported below in Section 5 support the conjecture that the H^1-norm does remain bounded in the dynamics of the $\mathcal{PT}$-symmetric Manakov system but may grow to infinity in the $\mathcal{PT}$-symmetric non-Manakov case with $g_{11} = g_{22} \neq g_{12}$.

3 Proof of Theorem 1

Consider a local solution

$$(u, v) \in C([-t_0, t_0], H^1(\mathbb{R}) \times H^1(\mathbb{R})) \tag{7}$$

to the generalized Manakov system (1), which exists for some $t_0 > 0$ due to the standard contraction mapping principle [16]. Let us recall the conserved quantities for the Hamiltonian version of the generalized Manakov equations (1) with $\gamma = 0$

$$Q(t) := \int_{\mathbb{R}} (|u|^2 + |v|^2) dx \tag{8}$$

and

$$E(t) := \int_{\mathbb{R}} \left(|u_x|^2 + |v_x|^2 + \kappa(\bar{u}v + u\bar{v}) \right.$$
$$\left. - \frac{g_{11}}{2}|u|^4 - \frac{g_{22}}{2}|v|^4 - g_{12}|u|^2|v|^2 \right) dx. \tag{9}$$

Both Q and E are well defined in the energy space, where the local solution (7) lives. If $\gamma \neq 0$, these integral quantities are no longer constant in time. Therefore, we shall first establish the balance equations for the rate of change of these quantities. In what follows, we use notations $\|u\|_{L^p}$ for the L^p-norm of $u : \mathbb{R} \mapsto \mathbb{C}$. If u also depends on t, we do not usually write the variable t explicitly.

From system (1), it is clear that the local solution (7) satisfies $(u_t, v_t) \in C([-t_0, t_0], H^{-1}(\mathbb{R}) \times H^{-1}(\mathbb{R}))$. Using the duality of Sobolev spaces $H^1(\mathbb{R})$ and $H^{-1}(\mathbb{R})$, we obtain

$$\frac{dQ}{dt} = 2\gamma \left(\|u\|_{L^2}^2 - \|v\|_{L^2}^2 \right) \leq 2\gamma Q(t). \tag{10}$$

Using Gronwall's inequality for (10), we deduce that $Q(t)$ does not blow up in a finite time and grows not faster than an exponential function:

$$Q(t) \leq Q(0)e^{2\gamma|t|}, \quad t \in [-t_0, t_0]. \tag{11}$$

The balance equation for the energy $E(t)$ is not defined if we only work with the local solution (7). However, we can approximate H^1-solutions of system (1) by sequences of local solutions of system (1) in $H^3(\mathbb{R})$, for which the energy balance equation is derived using integration by parts. Then, we take the limits for the sequences and obtain the following balance equation for the local solutions in $H^1(\mathbb{R})$:

$$\frac{dE}{dt} = 2\gamma \int_{\mathbb{R}} \left(|u_x|^2 - |v_x|^2 - g_{11}|u|^4 + g_{22}|v|^4 \right) dx. \tag{12}$$

We shall now control the homogeneous $H^1(\mathbb{R})$ norm of the solution by using the integral quantity

$$D(t) := \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2. \tag{13}$$

Note that $D(t)$ is a part of $E(t)$ but it does not satisfy a nice balance equation, for which the nonlinear terms cancel out. In other words, the energy balance equation for $D(t)$ cannot be estimated in terms of $D(t)$ only. For this purpose, we use $E(t)$ and, therefore, we shall first establish a correspondence between $D(t)$ and $E(t)$.

Let us recall the Gagliardo–Nirenberg inequality [16]. There exists a positive constant C_{GN} such that for every $f \in H^1(\mathbb{R})$, we have

$$\|f\|_{L^4}^4 \leq C_{GN} \|f\|_{L^2}^3 \|f_x\|_{L^2}. \tag{14}$$

Using the Gagliardo–Nirenberg inequality (14), we estimate $D(t)$ from above:

$$\begin{aligned} D(t) &= E(t) + \int_{\mathbb{R}} \left(\frac{g_{11}}{2}|u|^4 + \frac{g_{22}}{2}|v|^4 + g_{12}|u|^2|v|^2 - \kappa(\bar{u}v + u\bar{v}) \right) dx \\ &\leq |E(t)| + g \left(\|u\|_{L^4}^4 + \|v\|_{L^4}^4 \right) + \kappa Q(t) \\ &\leq |E(t)| + g C_{GN} Q(t)^{3/2} D(t)^{1/2} + \kappa Q(t), \end{aligned} \tag{15}$$

where $g := \max\{|g_{11}|, |g_{22}|, |g_{12}|\}$. Recall that both κ and γ are considered to be non-negative for simplicity. From inequality (15), we obtain

$$D(t)^{1/2} \leq \frac{g}{2} C_{GN} Q(t)^{3/2} + \sqrt{|E(t)| + \frac{g^2}{4} C_{GN}^2 Q(t)^3 + \kappa Q(t)}. \tag{16}$$

The energy balance (12) can now be estimated again with the use of the Gagliardo–Nirenberg inequality (14):

$$\begin{aligned} \left| \frac{dE}{dt} \right| &\leq 2\gamma \left(D(t) + g \left(\|u\|_{L^4}^4 + \|v\|_{L^4}^4 \right) \right) \\ &\leq 2\gamma \left(D(t) + g C_{GN} Q(t)^{3/2} D(t)^{1/2} \right), \end{aligned} \tag{17}$$

and using (15), (16), in order to bound $D(t)$, as well as the elementary inequality $2|ab| \leq |a|^2 + |b|^2$, we obtain

$$\left| \frac{dE}{dt} \right| \leq 4\gamma \left(|E(t)| + \kappa Q(t) + \frac{9}{8} g^2 C_{GN}^2 Q(t)^3 \right). \tag{18}$$

Using that $E(t) = E(0) + \int_0^t E'(\tau)d\tau$ and integrating (18), we obtain the integral inequality

$$|E(t)| \le |E(0)| + 4\gamma \int_0^t (|E(\tau)| + Z(\tau))\, d\tau, \tag{19}$$

where

$$Z(t) := \kappa Q(t) + \frac{9}{8} g^2 C_{GN}^2 Q(t)^3. \tag{20}$$

Notice that because of the bound (11), function $Z(t)$ is defined for all $t \in [-t_0, t_0]$ and does not grow faster than an exponential function. Using Gronwall's inequality for (19), we obtain

$$|E(t)| \le \left(|E(0)| + 4\gamma \int_0^t Z(\tau)d\tau \right) e^{4\gamma |t|}, \quad t \in [-t_0, t_0]. \tag{21}$$

Therefore $|E(t)|$ does not blow up in a finite time and grows not faster than an exponential function. The same is true for $D(t)$ thanks to the bound (16).

Applying the standard continuation arguments to extend the local solution of the generalized Manakov system (1) in $H^1(\mathbb{R})$ to larger values of t_0 by using a priori energy inequalities (11), (16), and (21), we conclude the proof of Theorem 1 on the existence of the unique global solution of the generalized Manakov system (1) in the energy space.

4 Proof of Theorem 2

For the global solution of Theorem 1, we can introduce the integral Stokes variables

$$S_1(t) := \int_{\mathbb{R}} (\bar{u}v + u\bar{v})dx,$$

$$S_2(t) := i \int_{\mathbb{R}} (\bar{u}v - u\bar{v})dx,$$

$$S_3(t) := \int_{\mathbb{R}} (|u|^2 - |v|^2)dx.$$

Let us consider the case $g_{11} = g_{22} = g_{12}$. We compute temporal derivatives of the integral Stokes variables and obtain

$$\frac{dS_1(t)}{dt} = 0, \quad \frac{dS_2(t)}{dt} = 2\kappa S_3, \quad \frac{dS_3(t)}{dt} = 2\gamma Q - 2\kappa S_2, \tag{22}$$

where Q is defined by (8) and satisfies the balance equation (10). Note that all nonlinear terms cancel out in the system of evolution equations (22) for the integral Stokes variables.

It follows from system (22) that S_1 and $C := \kappa Q - \gamma S_2$ are constants of motion. Excluding S_2 and S_3 from system (22), we obtain the linear oscillator equation for $Q(t)$

$$\frac{d^2 Q}{dt^2} + 4\omega^2 Q = 4\kappa C = \text{const.} \tag{23}$$

where we have introduced

$$\omega = \sqrt{\kappa^2 - \gamma^2} \tag{24}$$

The linear oscillator equation (23) immediately implies that if $\gamma < \kappa$, then $Q(t)$ oscillates with the frequency 2ω and therefore, can be globally bounded for all $t \in \mathbb{R}$ by a constant $Q_{max} > 0$. Indeed, from the linear equation (23) with the initial conditions $Q(0) = Q_0$ and $\dot{Q}(0) = 2\gamma S_3(0) = P_0$, we compute

$$Q(t) = \frac{\kappa C}{\omega^2} + A_1 \cos(2\omega t) + A_2 \sin(2\omega t), \tag{25}$$

where the integration constants A_1 and A_2 are given by

$$A_1 = Q_0 - \frac{\kappa^2 C}{\omega^2}, \quad \text{and} \quad A_2 = \frac{P_0}{2\omega}. \tag{26}$$

Hence $Q(t) \leq Q_{max}$, where the sharp upper bound Q_{max} is computed from the initial conditions by

$$Q_{max} = \frac{\kappa C}{\omega^2} + \sqrt{A_1^2 + A_2^2} \tag{27}$$

This argument proves bound (5) of Theorem 2.

If $\gamma \geq \kappa$, there exists solutions to (23) with $\omega^2 \leq 0$ such that $Q(t)$ grows unboundedly as $t \to \infty$. Therefore, the L^2-norm of global solutions of the $\mathcal{PT}$-symmetric Manakov system with $\gamma \geq \kappa$ grows to infinity, according to the limit (6). This concludes the proof of Theorem 2.

Note that the linear oscillator equation (23) is derived in our previous work [15] for the x-independent solutions by using the local Stokes variables. Furthermore, we note here that for x-independent solutions, the $\mathcal{PT}$-symmetric Manakov dimer (3) can be reduced to the linear system. Indeed, considering solutions of the system (3) in the form

$$u(t) = \tilde{u}(t)e^{i\varphi(t)}, \quad v(t) = \tilde{v}(t)e^{i\varphi(t)},$$

and setting $\varphi(t) = \int_0^t \left(|\tilde{u}|^2 + |\tilde{v}|^2 \right) dt$, we obtain the system of linear equations

$$\begin{aligned} i\tilde{u}_t &= \kappa\tilde{v} + i\gamma\tilde{u}, \\ i\tilde{v}_t &= \kappa\tilde{u} - i\gamma\tilde{v}. \end{aligned} \tag{28}$$

The linear system (28) is stable for $\gamma < \kappa$ and unstable for $\gamma \geq \kappa$. Although these results are rather trivial for x-independent solutions to the $\mathcal{PT}$-symmetric Manakov dimer (3), we have shown here that these results can be extended to bound the L^2-norm of x-dependent solutions to the $\mathcal{PT}$-symmetric Manakov system (1) with $g_{11} = g_{22} = g_{12}$ in the energy space.

5 Numerical Simulations

The results of Theorems 1 and 2 do not clarify if the H^1-norm of global solutions of the $\mathcal{PT}$-symmetric system (1) remains bounded for $\gamma < \kappa$. In order to obtain additional insight into this issue, we have undertaken a set of numerical simulations. Our numerical results indicate that the H^1-norm of the global solutions does remain bounded for the Manakov system ($g_{11} = g_{22} = g_{12}$) with unbroken $\mathcal{PT}$ symmetry ($\gamma < \kappa$). On the other hand, in the $\mathcal{PT}$-symmetric non-Manakov case with $g_{11} = g_{22} \neq g_{12}$, unbounded solutions exist even if $\gamma < \kappa$. The analytical proof of these conjectures remains opened for further studies.

More specifically, we run a set of numerical simulations of the dynamics governed by the generalized Manakov system (1) which is solved in the finite domain $[-l, l]$ with zero boundary conditions, $u(\pm l, t) = v(\pm l, t) = 0$, using a semi-implicit difference scheme. The numerically obtained solution is used to evaluate quantities $D(t)$ and $Q(t)$ which characterize the H^1-norm of the global solution. For the initial conditions at $t = 0$, we use Gaussian beams

$$u_0(x) = \frac{A}{\pi^{1/4}a^{1/2}} \exp\left\{-\frac{x^2}{2a^2}\right\}, \quad v_0(x) = \frac{B}{\pi^{1/4}b^{1/2}} \exp\left\{-\frac{x^2}{2b^2}\right\} \tag{29}$$

truncated on the interval $(-l, l)$, where A, B, a and b are real constants. Notice that the functions u_0 and v_0 are normalized such that

$$\int_{-\infty}^{\infty} |u_0(x)|^2 dx = A^2 \quad \text{and} \quad \int_{-\infty}^{\infty} |v_0(x)|^2 dx = B^2.$$

The width l of the computational domain is sufficiently large such that the numerical value of the initial conditions (29) nearly vanish at $x = \pm l$.

We solved the Cauchy problem in the finite domain $[-l, l]$ for several different sets of initials conditions (A, a, B, b), as well as for different combinations of γ and κ (with $\gamma < \kappa$) and for different l. In Fig. 1 we present results of numerical runs for three different initial conditions (shown in the left column). The quantities D and Q versus t are shown for the case of the Manakov nonlinearity with $g_{11} = g_{22} = g_{12} = 1$ (panels in the middle column) and the non-Manakov case with $g_{11} = g_{22} = 1$ and $g_{12} = 1/2$ (panels in the right column).

For all sets of the initial conditions (29) and for all l we have tested, we observed that in the case $g_{11} = g_{22} = g_{12}$, the numerically obtained H^1 norm remains bounded (see panels in the middle columns of Fig. 1). This allows us to conjecture on boundedness of the H^1-norm of all global solutions to the $\mathcal{PT}$-symmetric Manakov system (1) with $g_{11} = g_{22} = g_{12}$ and $\gamma < \kappa$. On the other hand, unbounded numerical solutions exist in the non-Manakov case (see the upper panel in the right column of Fig. 1). In all panels showing dynamics of the
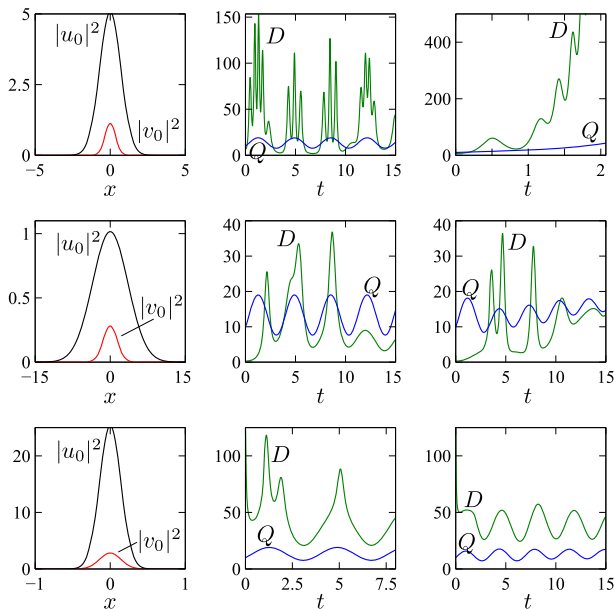

Fig. 1 Results of numerical solution of the Cauchy problem for $\mathcal{PT}$-symmetric system (1). Three rows present results obtained for different initial conditions (29): upper, middle and the lower rows correspond to (A, a, B, b) as follows: (3, 1, 1, 0.5), (3, 5, 1, 2), and (3, 0.2, 1, 0.2). Panels from the left column show profiles of the initial conditions $|u_0(x)|^2$ (*black lines*) and $|v_0(x)|^2$ (*red lines*). Panels from the central column show dependencies $D(t)$ (*green lines*, labeled as D) and $Q(t)$ (*blue lines*, labeled as Q) found for the Manakov case $g_{11} = g_{22} = g_{12} = 1$. Panels from the right column show the same data, obtained from the same initial conditions as in the middle column, but for the non-Manakov case $g_{11} = g_{22} = 1$ and $g_{12} = 1/2$. For all panels, $\kappa = 1$ and $\gamma = 1/2$, i.e., $\mathcal{PT}$ symmetry is unbroken

Manakov system, we observe the oscillatory behavior of $Q(t)$ which is characterized by the period (see (23) and (24)),

$$\frac{\pi}{\omega} = \frac{\pi}{\sqrt{\kappa^2 - \gamma^2}} \approx 3.6276. \tag{30}$$

Since our main interest is related to the existence (or non-existence) of globally bounded solutions, in Fig. 1 we show the results obtained for a situation when the input beam in the equation with gain has the intensity much larger than the intensity of the input beam for the NLS equation with dissipation (c.f. black and red curves in the left columns of Fig. 1). The opposite situation when the intensity of the input beam for the lossy equation is much larger than that for the equation with gain was also tested in our numerics with the same conclusions on boundedness and unboundedness of solutions.

6 Extension to the Two-Dimensional Manakov System

In this concluding section, we generalize the proof of the global existence in the energy space for the Manakov system, i.e. for the system (1) with all equal nonlinear coefficients, presented above, to the case of two spatial dimensions, $x \in \mathbb{R}^2$. In this context, we note that the analysis of global existence and blow-up of solutions in a Hamiltonian counterpart of this system, it corresponds to $\gamma = 0$, was recently presented in [18].

Respectively, we shall rewrite the system (1) in two spatial dimensions with the Manakov-type nonlinearity:

$$
\begin{aligned}
iu_t &= -\Delta u + \kappa v + i\gamma u - (|u|^2 + |v|^2)u, \\
iv_t &= -\Delta v + \kappa u - i\gamma v - (|u|^2 + |v|^2)v.
\end{aligned} \tag{31}
$$

Here Δ is the two-dimensional Laplacian and all the nonlinear coefficients g_{11}, g_{22} and g_{12} are set to unity.

We shall now formulate and prove the global existence result for the $\mathcal{PT}$-symmetric Manakov system (31) by generalizing the ideas of Theorems 1 and 2. This generalization is possible due to the fact that the L^2-norm of a global solution to system (31) is bounded for $\gamma < \kappa$ by the constant Q_{max}, which is determined from the initial conditions in Sobolev space $H^1(\mathbb{R}^2)$ by the explicit expression (27). This global bound allows us to obtain apriori bounds for the energy

$$E(t) := \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |\nabla v|^2 + \kappa(\bar{u}v + u\bar{v}) - \frac{1}{2}|u|^4 - \frac{1}{2}|v|^4 - |u|^2|v|^2 \right) dx.$$

Note here that the local solution in the energy space

$$(u(t), v(t)) \in C\left([-t_0, t_0], H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\right), \tag{32}$$

is not derived by the standard contraction mapping principle for the semi-linear equations. Nevertheless, a modified contraction method can be developed with the use of Stritcharz estimates as in [16]. Therefore, we use here the local solution (32) obtained from this modified contraction method.

Next, let us recall the Gagliardo-Nirenberg inequality in the space of two dimensions [16]. There exists a positive constant C_{GN} such that for every $f \in H^1(\mathbb{R}^2)$, we have

$$\|f\|_{L^4}^4 \le C_{GN} \|\nabla f\|_{L^2}^2 \|f\|_{L^2}^2. \tag{33}$$

Moreover, it was shown by Weinstein [19] that C_{GN} can be found sharply in terms of the ground state solution R of the stationary equation

$$\Delta R - R + R^3 = 0,$$

namely, $C_{GN} = 2/\|R\|_{L^2} \approx 0.171$. In physics, the ground state solution R is also known as the Townes soliton [20]. Now we can formulate a global existence result for the two-dimensional $\mathcal{PT}$-symmetric Manakov system (31).

Theorem 3 *Consider the two-dimensional $\mathcal{PT}$-symmetric Manakov system (31) with $\gamma < \kappa$. If the initial conditions $(u_0, v_0) \in H^1\left(\mathbb{R}^2\right) \times H^1\left(\mathbb{R}^2\right)$ satisfy the constraint*

$$Q_{max} < \frac{1}{2}\|R\|_{L^2}^2 \tag{34}$$

where Q_{max} is defined by (27), then there exists a unique global solution $(u(t), v(t)) \in C(\mathbb{R}, H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2))$ such that $u(0) = u_0$ and $v(0) = v_0$. On the other hand, if $Q_{max} \geq \|R\|_{L^2}^2$, there exists initial conditions $(u_0, v_0) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, such that local solutions (32) blow up in a finite time, that is, there exists $T > 0$ such that

$$\lim_{t \uparrow T} \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2 \right) = \infty. \tag{35}$$

We shall now sketch the proof of Theorem 3 by following the ideas of the proof of Theorem 1. Defining $D(t)$ by

$$D(t) := \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2, \tag{36}$$

we employ the Gagliardo-Nirenberg inequality (33) and estimate $D(t)$ from above as

$$\begin{aligned} D(t) &\leq |E(t)| + \|u\|_{L^4}^4 + \|v\|_{L^4}^4 + \kappa Q(t) \\ &\leq |E(t)| + C_{GN} D(t) Q(t) + \kappa Q(t). \end{aligned}$$

For all $t \in [-t_0, t_0]$ we have $Q(t) \leq Q_{max}$. If the condition (34) is met (recall that $C_{GN} = 2/\|R\|_{L^2}^2$), then $1 - C_{GN} Q_{max} > 0$, so that $D(t)$ is controlled by $|E(t)|$ from above as

$$D(t) \leq \frac{|E(t)| + \kappa Q_{max}}{1 - C_{GN} Q_{max}}. \tag{37}$$

Using then the balance equation for dE/dt as in (12), we obtain for all $t \in [-t_0, t_0]$,

$$\begin{aligned} \left| \frac{dE}{dt} \right| &\leq 2\gamma \left(D(t) + \|u\|_{L^4}^4 + \|v\|_{L^4}^4 \right) \\ &\leq 2\gamma \left(D(t) + C_{GN} D(t) Q(t) \right) \\ &\leq 2\gamma \frac{1 + C_{GN} Q_{max}}{1 - C_{GN} Q_{max}} \left(|E(t)| + \kappa Q_{max} \right). \end{aligned}$$

Integrating the latter inequality similarly to what is done in the derivation of (19) and (21), we obtain a global (exponentially growing) bound on the H^1-norm of the local solution. Hence, the local solution (32) can be extended to larger values of t_0, and finally, globally for all $t \in \mathbb{R}$.

To show the other part of Theorem 3 concerning the finite-time blow-up (35), we note the following trick, which is explored in the recent works [2, 7, 17]. Using the substitution

$v = e^{i\delta}u$ with δ defined by $\sin(\delta) = -\gamma/\kappa$, we reduce the $\mathcal{PT}$-symmetric Manakov system (31) with $\gamma < \kappa$ to a scalar NLS equation

$$iu_t = -\Delta u + \kappa\cos(\delta)u - 2|u|^2 u. \tag{38}$$

Solutions of (38) conserve the L^2-norm and may blow up in a finite time if $\|u_0\|_{L^2}^2 \geq \|R\|_{L^2}^2/2$ [19]. The corresponding blowing-up solution of the $\mathcal{PT}$-symmetric Manakov system (31) also conserves the L^2 norm, which then satisfies $Q(t) = Q_{max} = 2\|u_0\|_{L^2}^2 \geq \|R\|_{L^2}^2$. This concludes the proof of Theorem 3.

We note that Theorem 3 does not clarify if solutions to the $\mathcal{PT}$-symmetric Manakov system (31) may blow up in a finite time if the initial data corresponds to Q_{max} fitting in the interval

$$\frac{1}{2}\|R\|_{L^2}^2 \leq Q_{max} < \|R\|_{L^2}^2.$$

This question remains open for further studies.

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