DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **37**, Number **4**, April **2017**

doi:10.3934/dcds.2017088

pp. **2065–2075**

LONG-TIME STABILITY OF SMALL FPU SOLITARY WAVES

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(Communicated by Pierre Germain)

Abstract. Small-amplitude waves in the Fermi-Pasta-Ulam (FPU) lattice with weakly anharmonic interaction potentials are described by the generalized Korteweg-de Vries (KdV) equation. Justification of the small-amplitude approximation is usually performed on the time scale, for which dynamics of the KdV equation is defined. We show how to extend justification analysis on longer time intervals provided dynamics of the generalized KdV equation is globally well-posed in Sobolev spaces and either the Sobolev norms are globally bounded or they grow at most polynomially. The time intervals are extended respectively by the logarithmic or double logarithmic factors in terms of the small amplitude parameter. Controlling the approximation error on longer time intervals allows us to deduce nonlinear metastability of small FPU solitary waves from orbital stability of the KdV solitary waves.

1. **Introduction.** In this work, we address an open question from [6] on how to deduce nonlinear metastability or instability of small Fermi–Pasta–Ulam (FPU) solitary waves from orbital stability or instability of the Korteweg–de Vries (KdV) solitary waves. Let us consider dynamics of the FPU lattice given by Newton's equations of motion:

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}, \tag{1}$$

where $(x_n)_{n\in\mathbb{Z}}$ is a function of the time $t \in \mathbb{R}$, with values in $\mathbb{R}^{\mathbb{Z}}$, the dot denotes the time derivative, and the interaction potential V is smooth. The coordinate x_n corresponds to the displacement of the n-th particle in a one-dimensional chain from its equilibrium position. The potential V for anharmonic interactions of particles is taken in the form

$$V(u) = \frac{1}{2}u^2 + \frac{\epsilon^2}{p+1}u^{p+1}, \tag{2}$$

2010 *Mathematics Subject Classification.* Primary: 37K60; Secondary: 35Q53, 37K40.

Key words and phrases. Fermi–Pasta–Ulam lattice, Korteweg–de Vries equations, stability of solitary waves, small-amplitude approximations, long-time stability.

The work of A. Khan was performed during MSc program at McMaster University in 2013-2015. The work of D.E. Pelinovsky is supported by the NSERC grant. The authors thank E. Dumas, T. Penati, and G. Schneider for discussions and collaborations.

where $p \geq 2$ is integer and the strength of anharmonicity ϵ can be introduced by the scaling transformation. The FPU lattice equations (1) can be rewritten in the strain variables $u_n := x_{n+1} - x_n$ as follows

$$\ddot{u}_n = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z}. \tag{3}$$

Using the well-known asymptotic multi-scale expansion [2, 7, 21],

$$u_n(t) = W(\epsilon(n-t), \epsilon^3 t) + \text{error terms}, \tag{4}$$

yields the generalized KdV equation for the leading-order approximation W given by

$$2W_\tau + \frac{1}{12} W_{\xi\xi\xi} + (W^p)_\xi = 0, \quad \xi \in \mathbb{R}, \tag{5}$$

where $\tau = \epsilon^3 t$ and $\xi = \epsilon(n-t)$.

Local well-posedness of the generalized KdV equation (5) in Sobolev spaces $H^s(\mathbb{R})$ is known from the works of Kato [10, 11] for $s > \frac{3}{2}$ and Kenig–Ponce–Vega [12, 13] for $s \geq s_p$, where

$$s_{p=2} = \frac{3}{4}, \quad s_{p=3} = \frac{1}{4}, \quad s_{p=4} = \frac{1}{12}, \quad s_{p\geq 5} = \frac{p-5}{2(p-1)}.$$

For any local solution $W \in C([-\tau_0, \tau_0], H^s(\mathbb{R}))$ of the KdV equation (5) with $s \geq 6$ and $\tau_0 > 0$, the error terms in the asymptotic multi-scale expansion (4) can be controlled as follows. There exist positive constants ϵ_0 and C_0 such that, for all $\epsilon \in (0, \epsilon_0)$, when initial data $(u_{\text{in}}, \dot{u}_{\text{in}}) \in \ell^2(\mathbb{R})$ satisfy

$$\|u_{\text{in}} - W(\epsilon \cdot, 0)\|_{\ell^2} + \|\dot{u}_{\text{in}} + \epsilon \partial_\xi W(\epsilon \cdot, 0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{6}$$

the unique solution $(u, \dot{u})$ to the FPU equation (3) with initial data $(u_{\text{in}}, \dot{u}_{\text{in}})$ belongs to $C^1([-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}], \ell^2(\mathbb{Z}))$ and satisfies

$$\|u(t) - W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2} + \|\dot{u}(t) + \epsilon \partial_\xi W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2} \leq C_0 \epsilon^{3/2}, \tag{7}$$

for every $t \in \left[-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}\right]$. The proof of this result is based on the energy estimates and Gronwall's inequality [2, 7, 21].

Bound (7) suggests that small-amplitude FPU solitary waves are metastable or unstable if the KdV solitary waves are orbitally stable or unstable. Indeed, the generalized KdV equation (5) is known to have orbitally stable solitary waves for $p = 2, 3, 4$ and orbitally unstable solitary waves for $p \geq 5$ [18]. However, this simple and widely accepted analogy appears in apparent contradiction with the energy arguments found in [6] suggesting unconditional metastability of small-amplitude FPU solitary waves on the time scale of $\mathcal{O}(\epsilon^{-3})$.

The metastability result from [6] can be formulated as follows. Let us denote the traveling-wave solutions of the FPU equation (3) by u_{trav}. Then, for every $\tau_0 > 0$, there exist positive constants ϵ_0, δ_0 and C_0 such that, for all $\epsilon \in (0, \epsilon_0)$, when initial data $(u_{\text{in}}, \dot{u}_{\text{in}}) \in \ell^2(\mathbb{R})$ satisfy

$$\delta := \|u_{\text{in}} - u_{\text{trav}}(0)\|_{\ell^2} + \|\dot{u}_{\text{in}} - \dot{u}_{\text{trav}}(0)\|_{\ell^2} \leq \delta_0, \tag{8}$$

the unique solution $(u, \dot{u})$ to the FPU equation (3) with initial data $(u_{\text{in}}, \dot{u}_{\text{in}})$ belongs to $C^1([-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}], \ell^2(\mathbb{Z}))$ and satisfies

$$\|u(t) - u_{\text{trav}}(t)\|_{l^2} + \|\dot{u}(t) - \dot{u}_{\text{trav}}(t)\|_{l^2} \leq C_0 \delta, \tag{9}$$

for every $t \in \left[-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}\right]$. Similarly to the bound (7), the bound (9) is also proved with the energy estimates and Gronwall's inequality complemented with the asymptotic scaling of small-amplitude FPU traveling waves u_{trav} near the KdV solitary waves [6]. Nevertheless, the initial data $(u_{\rm in}, \dot{u}_{\rm in})$ can be modulated on any spatial scale.

Bound (9) suggests unconditional metastability of small-amplitude FPU solitary waves up to the time scale of $\mathcal{O}(\epsilon^{-3})$ for every $p \geq 2$. This may be viewed as a contradiction with the bound (7) that suggests instability of small-amplitude FPU solitary waves at the time scale of $\mathcal{O}(\epsilon^{-3})$ for $p \geq 5$ because the corresponding KdV solitary waves are unstable for $p \geq 5$ [18].

Of course, no contradiction arises because the energy methods used in the proof of the upper bounds on the approximation errors (7) and (9) yield constants C_0 that grow exponentially in time τ_0, that is, on the time scale of $\mathcal{O}(\epsilon^{-3})$. As a result, the exponential divergence of the constant C_0 cannot be distinguished from the exponential instability of the KdV solitary waves in the case $p \geq 5$. However, this observation also shows that the bound (9) is not a reliable evidence to conclude on metastability of the small-amplitude FPU solitary waves in the case of $p = 2, 3, 4$.

Nonlinear stability of small-amplitude FPU solitary waves in the case of the classical KdV equation with $p = 2$ was studied in the series of papers by Friesecke & Pego [7] based on the orbital and asymptotic stability of the KdV solitons [19]. Similarly, asymptotic stability of several solitary waves was studied by Mizumachi [16, 17] and Benes, Hoffmann & Wayne [3, 9] also in the case $p = 2$. Derivation and analysis of small-amplitude FPU solitary waves were recently generalized for polyatomic FPU lattices in [8].

In the present work, we extend the bound (7) on the approximation error to longer time intervals provided dynamics of the generalized KdV equation (5) is globally well-posed in Sobolev spaces and either the Sobolev norms are globally bounded or they grow at most polynomially.

For the integrable cases $p = 2$ and $p = 3$, a uniform bound on the $H^s(\mathbb{R})$ norms for any $s \in \mathbb{N}$ can be obtained from conserved quantities of the KdV and modified KdV hierarchies [5, 15]. For the non-integrable case $p = 4$, the global solution is controlled in $H^1(\mathbb{R})$ by using the energy conservation, while the $H^s(\mathbb{R})$ norms with $s \geq 2$ can grow at most polynomially. In particular, it was proved by Staffilani [22] that for any $s \geq 2$, there exists a constant C_s such that the unique solution of the generalized KdV equation (5) with $p = 4$ satisfies

$$\|W(\tau)\|_{H^s} \leq C_s |\tau|^{s-1} \quad \text{as} \quad |\tau| \to \infty. \tag{10}$$

Global solutions to the generalized KdV equation (5) with $p \geq 5$ exist and scatter to zero [13] if the $H^{s_p}(\mathbb{R})$ norm of initial data is small, where

$$s_p = \frac{p-5}{2(p-1)}, \quad p \geq 5. \tag{11}$$

For the global solutions scattering to zero in the case $p \geq 5$, the $H^1(\mathbb{R})$ norm is again controlled by the energy conservation [13], and the polynomial bound (10) holds [22].

The following theorems extend the bound (7) on the approximation error to longer time intervals. The two cases have to be considered separately, depending whether the $H^s(\mathbb{R})$ norm of the KdV solution is globally bounded or may grow at most polynomially.

Theorem 1. *Let $W \in C(\mathbb{R}, H^s(\mathbb{R}))$ be a global solution to the generalized KdV equation (5) with either $p = 2$ or $p = 3$ for some integer $s \geq 6$. For fixed $r \in \left(0, \frac{1}{2}\right)$, there exist positive constants ϵ_0, C, and K such that, for all $\epsilon \in (0, \epsilon_0)$, when initial* *data* $(u_{\rm in}, \dot{u}_{\rm in}) \in \ell^2(\mathbb{R})$ *satisfy*

$$\|u_{\rm in} - W(\epsilon\cdot, 0)\|_{\ell^2} + \|\dot{u}_{\rm in} + \epsilon\partial_\xi W(\epsilon\cdot, 0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{12}$$

the unique solution $(u, \dot{u})$ *to the FPU equation (3) with initial data* $(u_{\rm in}, \dot{u}_{\rm in})$ *belongs to* $C^1([-t_0(\epsilon), t_0(\epsilon)], \ell^2(\mathbb{Z}))$ *with* $t_0(\epsilon) := rK^{-1}\epsilon^{-3}|\log(\epsilon)|$ *and satisfies*

$$\|u(t) - W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2} + \|\dot{u}(t) + \epsilon\partial_\xi W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2} \leq C\epsilon^{3/2-r}, \tag{13}$$

for every $t \in [-t_0(\epsilon), t_0(\epsilon)]$.

Theorem 2. *Let* $W \in C(\mathbb{R}, H^s(\mathbb{R}))$ *be a global solution to the generalized KdV equation* (5) *with either* $p = 4$ *or* $p \geq 5$ *and small* $\|W(0)\|_{H^{s_p}}$, *for some integer* $s \geq 6$. *For fixed* $r \in \left(0, \frac{1}{2}\right)$, *there exist positive constants* ϵ_0, C, *and* K *such that, for all* $\epsilon \in (0, \epsilon_0)$, *when initial data* $(u_{\rm in}, \dot{u}_{\rm in}) \in \ell^2(\mathbb{R})$ *satisfy*

$$\|u_{\rm in} - W(\epsilon\cdot, 0)\|_{\ell^2} + \|\dot{u}_{\rm in} + \epsilon\partial_\xi W(\epsilon\cdot, 0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{14}$$

the unique solution $(u, \dot{u})$ *to the FPU equation (3) with initial data* $(u_{\rm in}, \dot{u}_{\rm in})$ *belongs to* $C^1([-t_0(\epsilon), t_0(\epsilon)], \ell^2(\mathbb{Z}))$ *with* $t_0(\epsilon) := (2pK)^{-1}\epsilon^{-3}\log(r|\log(\epsilon)|)$ *and satisfies*

$$\|u(t) - W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2} + \|\dot{u}(t) + \epsilon\partial_\xi W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2} \leq C\epsilon^{3/2-r}, \tag{15}$$

for every $t \in [-t_0(\epsilon), t_0(\epsilon)]$.

Remark 1. The final time of the dynamics of the generalized KdV equation (5) given by $\tau_0(\epsilon) := \epsilon^3 t_0(\epsilon)$ depends on ϵ and satisfies $\tau_0(\epsilon) \to \infty$ as $\epsilon \to 0$ both in Theorems 1 and 2.

Remark 2. The fixed parameter $r \in \left(0, \frac{1}{2}\right)$ in Theorems 1 and 2 determines either the logarithmic or the double-logarithmic factor in the extended time scale $t_0(\epsilon)$. It also determines the price to be paid for extending the KdV approximation to such long times in bounds (13) and (15). For the natural time scale of the generalized KdV equation, where $t_0(\epsilon) = \mathcal{O}(\epsilon^{-3})$, the approximation errors are of the $\mathcal{O}(\epsilon^{3/2})$ order in the $\ell^2(\mathbb{Z})$ norm, as it can be seen from (7).

Remark 3. Bounds (13) and (15) allow us to deduce nonlinear metastability of small FPU solitary waves in the solution $(u, \dot{u})$ from orbital stability of the KdV solitary waves in the solution W to the generalized KdV equation (5). In particular, solitary waves of the generalized KdV equation (5) are orbitally stable for $p = 2, 3, 4$, and so are small-amplitude FPU solitary waves on long but finite time intervals.

Remark 4. Solitary waves of the generalized KdV equation (5) are unstable for $p \geq 5$ and the class of global solutions considered in Theorem 2 for $p \geq 5$ excludes solitary waves. It follows from the bound (15) that dynamics of the small-amplitude waves in the FPU lattice resembles scattering dynamics of small solutions to the generalized KdV equation (5) with $p \geq 5$ [13].

Extended approximations on longer time intervals become increasingly popular in the justification analysis of amplitude equations in various evolutionary problems. One of the pioneer works is developed by Lannes & Rauch in the context of validity of the nonlinear geometric optics equations [14]. Extended time intervals modified by a logarithmic factor of ϵ were introduced in the justification of the discrete nonlinear Schrödinger equation in the context of the FPU lattices [4] and the Klein–Gordon lattices [20]. Our work addresses the extended time intervals in the justification of the KdV equation in the context of the FPU lattices.

The rest of the paper is structured as follows. Section 2 represents the basic set up for justification analysis of the generalized KdV equation (5) from the FPU lattice equation (3). Justification arguments on the KdV time scale are well-known and follow the formalism described in [21] with a refinement given in [6]. Sections 3 and 4 present details of the proofs of Theorems 1 and 2. This part is original and represents the main result of this paper.

2. **Justification setup.** The scalar second-order equation (3) can be rewritten as the following first-order evolution system

$$\left\{ \begin{array}{l} \dot{u}_n = q_{n+1} - q_n, \\ \dot{q}_n = u_n - u_{n-1} + \epsilon^2 (u_n^p - u_{n-1}^p), \end{array} \right. \qquad n \in \mathbb{Z}. \tag{16}$$

Local solutions $(u,q) \in C^1([-t_0,t_0],\ell^2(\mathbb{Z}))$ exist by standard Picard iterations, thanks to analyticity of the power nonlinearity with $p \in \mathbb{N}$ and to the boundness of the shift operators on $\ell^2(\mathbb{Z})$. For a given initial data $(u_{\rm in}, q_{\rm in}) \in \ell^2(\mathbb{Z})$, local solutions are extended to the global solutions $(u,q) \in C^1(\mathbb{R},\ell^2(\mathbb{Z}))$ by decreasing the values of ϵ thanks to the energy

$$H = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(q_n^2 + u_n^2 + \frac{2\epsilon^2}{p+1} u_n^{p+1} \right), \tag{17}$$

which conserves in time t. If p is odd, no constraints on ϵ arise for existence of global solutions $(u,q) \in C^1(\mathbb{R},\ell^2(\mathbb{Z}))$ to the FPU lattice equations (16).

Let us use the decomposition

$$\left\{ \begin{array}{l} u_n(t) = W(\epsilon(n-t),\epsilon^3 t) + \mathcal{U}_n(t), \\ q_n(t) = P_\epsilon(\epsilon(n-t),\epsilon^3 t) + \mathcal{Q}_n(t), \end{array} \right. \qquad n \in \mathbb{Z}, \tag{18}$$

where $W(\xi,\tau)$ is a suitable solution to the generalized KdV equation (5) (and thus W is ϵ-independent), whereas the ϵ-dependent function $P_\epsilon(\xi,\tau)$ is found from the first equation of the system (16) rewritten as

$$P_\epsilon(\xi+\epsilon,\tau) - P_\epsilon(\xi,\tau) = -\epsilon \partial_\xi W(\xi,\tau) + \epsilon^3 \partial_\tau W(\xi,\tau). \tag{19}$$

Looking for an approximate solution P_ϵ to this equation up to and including the formal order of $\mathcal{O}(\epsilon^3)$, we write

$$P_\epsilon := P^{(0)} + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + \epsilon^3 P^{(3)} \tag{20}$$

and collect the corresponding powers of ϵ. After routine computations (see, e.g., [6]), we obtain

$$\left\{ \begin{array}{l} P^{(0)} = -W, \\ P^{(1)} = \frac{1}{2}\partial_\xi W, \\ P^{(2)} = -\frac{1}{8}\partial_\xi^2 W - \frac{1}{2} W^p, \\ P^{(3)} = \frac{1}{48}\partial_\xi^3 W + \frac{1}{4} p W^{p-1} \partial_\xi W. \end{array} \right. \tag{21}$$

Note that equation (19) is satisfied by the expansion (20) with (21) only approximately, up to the terms of the formal order $\mathcal{O}(\epsilon^5)$.

Functions W and P_ϵ depend on $\xi = \epsilon(n-t)$. In order to be able to control the residual terms of the KdV approximation, we will use the following lemma proved in [6] (see also [21] for a weaker result). Note that a similar result is obtained in [1] in the subspace of $H^1(\mathbb{R})$ spanned by the piecewise linear functions (and with the constant $C = 1$).

Lemma 1. *There exists $C > 0$ such that for all $X \in H^1(\mathbb{R})$ and $\epsilon \in (0,1)$,*

$$\|x\|_{\ell^2} \le C\epsilon^{-1/2}\|X\|_{H^1},$$

where $x_n := X(\epsilon n)$, $n \in \mathbb{Z}$.

Substituting the decompositions (18), (20), and (21) into the FPU lattice equations (16), we obtain the evolution problem for the error terms

$$\begin{cases} \dot{\mathcal{U}}_n = \mathcal{Q}_{n+1} - \mathcal{Q}_n + \mathrm{Res}_n^{(1)}(t), \\ \dot{\mathcal{Q}}_n = \mathcal{U}_n - \mathcal{U}_{n-1} + \mathcal{R}_n(W,\mathcal{U}) + \mathrm{Res}_n^{(2)}(t) \\ \qquad + p\epsilon^2 W^{p-1}(\epsilon(n-t),\epsilon^3 t)\mathcal{U}_n - p\epsilon^2 W^{p-1}(\epsilon(n-1-t),\epsilon^3 t)\mathcal{U}_{n-1}, \end{cases} \tag{22}$$

where the residual and nonlinear terms are given by

$$\begin{aligned} \mathrm{Res}_n^{(1)}(t) \quad &:= \quad \epsilon\partial_\xi W(\epsilon(n-t),\epsilon^3 t) - \epsilon^3\partial_\tau W(\epsilon(n-t),\epsilon^3 t) \\ &\qquad + P_\epsilon(\epsilon(n+1-t),\epsilon^3 t) - P_\epsilon(\epsilon(n-t),\epsilon^3 t), \\ \mathrm{Res}_n^{(2)}(t) \quad &:= \quad \epsilon\partial_\xi P_\epsilon(\epsilon(n-t),\epsilon^3 t) - \epsilon^3\partial_\tau P_\epsilon(\epsilon(n-t),\epsilon^3 t) \\ &\qquad + W(\epsilon(n-t),\epsilon^3 t) - W(\epsilon(n-1-t),\epsilon^3 t) \\ &\qquad + \epsilon^2\left[W^p(\epsilon(n-t),\epsilon^3 t) - W^p(\epsilon(n-1-t),\epsilon^3 t)\right] \end{aligned}$$

and

$$\mathcal{R}_n(W,\mathcal{U})(t) := \epsilon^2 \sum_{k=2}^{p} \begin{pmatrix} p \\ k \end{pmatrix} \left[W^{p-k}(\epsilon(n-t),\epsilon^3 t)\mathcal{U}_n^k - W^{p-k}(\epsilon(n-1-t),\epsilon^3 t)\mathcal{U}_{n-1}^k\right].$$

Let us assume that the function $W(\epsilon(\cdot - t),\epsilon^3 t)$ belongs to $H^s(\mathbb{R})$ for $s \ge 6$ and satisfies the generalized KdV equation (5). The requirement $s \ge 6$ is needed to control the fifth derivative of W among the residual terms of the evolution problem. The following result provide bounds on the residual and nonlinear terms.

Lemma 2. *Let $W \in C([-\tau_0,\tau_0],H^s(\mathbb{R}))$ be a solution to the generalized KdV equation (5), for an integer $s \ge 6$ and $\tau_0 > 0$. Define*

$$\delta := \sup_{\tau\in[-\tau_0,\tau_0]} \|W(\tau)\|_{H^s}. \tag{23}$$

There exists a positive δ-independent constant C such that the residual and nonlinear terms satisfy

$$\|\mathrm{Res}^{(1)}(t)\|_{\ell^2} + \|\mathrm{Res}^{(2)}(t)\|_{\ell^2} \le C\left(\delta + \delta^{2p-1}\right)\epsilon^{9/2} \tag{24}$$

and

$$\|\mathcal{R}(W,\mathcal{U})(t)\|_{\ell^2} \le \epsilon^2 C\left(\delta^{p-2} + \|\mathcal{U}\|_{\ell^2}^{p-2}\right)\|\mathcal{U}\|_{\ell^2}^2 \tag{25}$$

for every $t \in [-\tau_0\epsilon^{-3},\tau_0\epsilon^{-3}]$ and $\epsilon \in (0,1)$.

Proof. By constructing P_ϵ in (20) and (21), we have canceled all terms in $\mathrm{Res}^{(1)}(t)$ up to and including the formal order ϵ^4. The remainder terms can be written in the closed form with Taylor's theorem as follows:

$$\epsilon^5\int_0^1 (1-r)^4\partial_\xi^5 W(\epsilon(n-t+r),\epsilon^3 t)\mathrm{d}r \tag{26}$$

and

$$\epsilon^5\int_0^1 (1-r)^2\partial_\xi^3 W^p(\epsilon(n-t+r),\epsilon^3 t)\mathrm{d}r. \tag{27}$$

The associated $\ell^2(\mathbb{Z})$ norm is estimated by $\left(\|W\|_{H^6} + \|W\|_{H^6}^p\right)\epsilon^{9/2}$, thanks to Lemma 1.

By formal expansion of $\mathrm{Res}^{(2)}(t)$ in ϵ, we confirm that all terms up to and including the formal order ϵ^4 are canceled if W satisfies the generalized KdV equation (5). On the other hand, the remainder terms contains terms like (26), (27), and additionally terms like

$$\epsilon^5 \int_0^1 \partial_\xi W^{2p-1}(\epsilon(n-t+r), \epsilon^3 t)\mathrm{d}r. \tag{28}$$

The associated $\ell^2(\mathbb{Z})$ norm is estimated by $\left(\|W\|_{H^6} + \|W\|_{H^6}^p + \|W\|_{H^6}^{2p-1}\right)\epsilon^{9/2}$, thanks to Lemma 1. Hence, we obtain the bound (24) by interpolating between the end point terms.

To prove the bound (25), we interpolate the binomial expansion for $\mathcal{R}_n(W,\mathcal{U})(t)$ between the end point terms and obtain for some $C > 0$:

$$\|\mathcal{R}(W,\mathcal{U})(t)\|_{\ell^2} \le C\epsilon^2 \left(\|W(\epsilon(\cdot - t), \epsilon^3 t))\|_{L^\infty}^{p-2}\|\mathcal{U}\|_{\ell^2}^2 + \|\mathcal{U}\|_{\ell^p}^p\right),$$

where we have used the bound $\|\mathcal{U}\|_{\ell^q} \le \|\mathcal{U}\|_{\ell^s}$ for every $1 \le s \le q \le \infty$. By using continuous embeddings of $H^6(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and $\ell^2(\mathbb{Z})$ to $\ell^\infty(\mathbb{Z})$, we obtain the bound (25). $\square$

For a local solution $(\mathcal{U}, \mathcal{Q}) \in C^1([-t_0, t_0], \ell^2(\mathbb{Z}))$ with some $t_0 > 0$ to the perturbed FPU lattice equations (22), we define the energy-type quantity

$$\mathcal{E}(t) := \frac{1}{2}\sum_{n\in\mathbb{Z}} \left[\mathcal{Q}_n^2(t) + \mathcal{U}_n^2(t) + \epsilon^2 p W^{p-1}(\epsilon(n-t), \epsilon^3 t))\mathcal{U}_n^2(t)\right]. \tag{29}$$

The following lemma describes properties of the energy-type quantity $\mathcal{E}(t)$.

Lemma 3. *Let $W \in C([-\tau_0, \tau_0], H^s(\mathbb{R}))$ be a solution to the generalized KdV equation (5), for an integer $s \ge 6$ and $\tau_0 > 0$. Let $\epsilon_0 > 0$ be defined by*

$$\epsilon_0 := \min\left\{1, (2p)^{-1/2}\left(\sup_{\tau\in[-\tau_0,\tau_0]} \|W(\cdot,\tau)\|_{L^\infty}\right)^{-(p-1)/2}\right\} \tag{30}$$

For every $\epsilon \in (0, \epsilon_0)$ and for every local solution $(\mathcal{U}, \mathcal{Q}) \in C^1([-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}], \ell^2(\mathbb{Z}))$ to system (22), the energy-type quantity (29) is coercive with the bound

$$\|\mathcal{Q}(t)\|_{l^2}^2 + \|\mathcal{U}(t)\|_{\ell^2}^2 \le 4\mathcal{E}(t), \quad t \in (-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}). \tag{31}$$

Moreover, when δ is defined by (23), there exists a positive (ϵ,δ)-independent constant C such that

$$\left|\frac{d\mathcal{E}}{dt}\right| \le C\mathcal{E}^{1/2}\left[(\delta + \delta^{2p-1})\epsilon^{9/2} + \epsilon^3(\delta^{p-1} + \delta^{2p-2})\mathcal{E}^{1/2} + \epsilon^2(\delta^{p-2} + \mathcal{E}^{(p-2)/2})\mathcal{E}\right],$$

for every $t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]$ and $\epsilon \in (0, \epsilon_0)$.

Proof. Coercivity (31) follows from the lower bound applied to (29)

$$2\mathcal{E}(t) \ge \|\mathcal{Q}(t)\|_\ell^2 + \left(1 - \epsilon^2 p\|W(\cdot,\tau)\|_{L^\infty}^{p-1}\right)\|\mathcal{U}(t)\|_{\ell^2}^2.$$

Since $1 - \epsilon^2 p\|W(\cdot,\tau)\|_{L^\infty}^{p-1} \ge 1/2$, we obtain the bound (31). Note that if p is odd, the lower bound for (29) implies (31) with $2\mathcal{E}(t)$ replacing $4\mathcal{E}(t)$. In this case, no constraint on ϵ is needed for the coercivity of $\mathcal{E}(t)$ but the constraint (30) is still used in the estimates of $\mathcal{E}'(t)$ as follows.

Taking derivative of $\mathcal{E}$ with respect to time t and using the perturbed FPU lattice equations (22) yield the evolution of the energy-type quantity:

$$\begin{aligned}\frac{d\mathcal{E}}{dt} = \sum_{n\in\mathbb{Z}} \Big[& \mathcal{Q}_n(t)\mathcal{R}_n(W,\mathcal{U})(t) + \mathcal{Q}_n(t)\mathrm{Res}_n^{(2)}(t) \\ & + \mathcal{U}_n(t)\left[1+\epsilon^2 p W^{p-1}(\epsilon(n-t),\epsilon^3 t)\right]\mathrm{Res}_n^{(1)}(t) \\ & + \frac{1}{2}\epsilon^2 p(p-1)W^{p-2}(\epsilon(n-t),\epsilon^3 t)\mathcal{U}_n^2(t)(-\epsilon\partial_\xi + \epsilon^3\partial_\tau)W(\epsilon(n-t),\epsilon^3 t)\Big].\end{aligned}$$

By using the Cauchy–Schwarz inequality, we estimate

$$\begin{aligned}\left|\frac{d\mathcal{E}}{dt}\right| \le\ & \|\mathcal{Q}\|_{\ell^2}\|\mathcal{R}(W,\mathcal{U})\|_{\ell^2} + \|\mathcal{Q}\|_{\ell^2}\|\mathrm{Res}^{(2)}\|_{\ell^2} + \frac{3}{2}\|\mathcal{U}\|_{\ell^2}\left\|\mathrm{Res}^{(1)}\right\|_{l^2} \\ & + \frac{1}{2}\epsilon^3 p(p-1)\|W\|_{L^\infty}^{p-2}\left(\|\partial_\xi W(\cdot,\tau)\|_{L^\infty} + \epsilon^2\|\partial_\tau W(\cdot,\tau)\|_{L^\infty}\right)\|\mathcal{U}(t)\|_{\ell^2}^2.\end{aligned}$$

By using estimates (24) and (25) in Lemma 2, the generalized KdV equation (5) for W, and the coercivity bound (31), we finally obtain the estimate on $\mathcal{E}'(t)$ asserted in the lemma. □

3. **Proof of Theorem 1.** Here we use the formalism of Section 2 and prove Theorem 1.

We consider global solutions to the KdV and modified KdV equations ($p = 2, 3$). In this case, it is known [5] that for any global solution $W \in C(\mathbb{R}, H^s(\mathbb{R}))$ with an integer $s \ge 6$, there exists a positive constant δ that only depends on the initial value of W at $\tau = 0$ such that

$$\|W(\cdot,\tau)\|_{H^s} \le \delta \quad \text{for every} \quad \tau \in \mathbb{R}. \tag{32}$$

In what follows, we neglect mentioning that all constants depend on the choice of $s \ge 6$.

For any initial data $(u_{\mathrm{in}}, \dot{u}_{\mathrm{in}}) \in \ell^2(\mathbb{R})$ satisfying the bound (12), there exists a local solution $(u, \dot{u}) \in C^1((-t_0, t_0), \ell^2(\mathbb{Z}))$ to the FPU equation (3). Equivalently, there exists a local solution $(u, q) \in C^1((-t_0, t_0), \ell^2(\mathbb{Z}))$ to the FPU lattice equations (16). The solution can be decomposed according to equation (18).

Let us set $\mathcal{S} := \mathcal{E}^{1/2}$, where $\mathcal{E}(t)$ is the energy-type quantity defined by (29). The initial bound (12) ensures that $\mathcal{S}(0) \le C_0\epsilon^{3/2}$ for some constant $C_0 > 0$, if ϵ_0 is chosen by (30). For fixed ϵ-independent constants $r \in \left(0, \frac{1}{2}\right)$, $C > C_0$, and $K > 0$, let us define the maximal continuation time

$$T_{C,K,r} := \sup\left\{T_0 \in (0, rK^{-1}\epsilon^{-3}|\log(\epsilon)|] : \ \mathcal{S}(t) \le C\epsilon^{3/2-r}, \ t \in [-T_0, T_0]\right\}. \tag{33}$$

Let us also define the corresponding evolution time of the generalized KdV equation (5) by

$$\tau_0(\epsilon) = rK^{-1}|\log(\epsilon)|.$$

Note that although the KdV and modified KdV equations are globally well-posed, we only consider their time evolution on a finite time interval, according to (33). The energy estimate of Lemma 3 can be rewritten for the variable $\mathcal{S}$ by

$$\left|\frac{d\mathcal{S}}{dt}\right| \le C_1(\delta + \delta^{2p-1})\epsilon^{9/2} + \epsilon^3 C_2\left[(\delta^{p-1} + \delta^{2p-2}) + \epsilon^{-1}(\delta^{p-2} + \mathcal{S}^{p-2})\mathcal{S}\right]\mathcal{S}, \tag{34}$$

where C_1 and C_2 are positive constants that are independent of δ and ϵ. Since δ is independent of $\tau_0(\epsilon)$ and hence is independent of ϵ, we can choose an ϵ-independent positive constant K sufficiently large such that

$$C_2\left[(\delta^{p-1}+\delta^{2p-2})+\epsilon^{-1}(\delta^{p-2}+C^{p-2}\epsilon^{(3/2-r)(p-2)})C\epsilon^{3/2-r}\right]\leq K, \tag{35}$$

as long as $\mathcal{S}(t)\leq C\epsilon^{3/2-r}$, since $r\in\left(0,\frac{1}{2}\right)$. Using (34) and (35), we obtain

$$\left|\frac{d}{dt}e^{-\epsilon^3 Kt}\mathcal{S}\right|\leq C_1(\delta+\delta^{2p-1})\epsilon^{9/2}e^{-\epsilon^3 Kt}. \tag{36}$$

By Gronwall's inequality, we obtain

$$\mathcal{S}(t)\leq\left(\mathcal{S}(0)+K^{-1}C_1(\delta+\delta^{2p-1})\epsilon^{3/2}\right)e^{K\tau_0(\epsilon)},\quad t\in[-T_{C,K,r},T_{C,K,r}], \tag{37}$$

where the exponent is extended to the maximal continuation time $t_0(\epsilon):=\epsilon^{-3}\tau_0(\epsilon)$. From the definition of $\tau_0(\epsilon)$, we have

$$\tau_0(\epsilon)=rK^{-1}|\log(\epsilon)|\quad\Rightarrow\quad e^{K\tau_0(\epsilon)}=\epsilon^{-r}.$$

Thus, we get from (37),

$$\mathcal{S}(t)\leq\left(C_0+K^{-1}C_1(\delta+\delta^{2p-1})\right)\epsilon^{3/2-r},\quad t\in[-T_{C,K,r},T_{C,K,r}]. \tag{38}$$

One can choose an ϵ-independent constant $C>C_0$ sufficiently large such that

$$C_0+K^{-1}C_1(\delta+\delta^{2p-1})\leq C. \tag{39}$$

Under the constraints (35) and (39) on $C>C_0$ and $K>0$, the time interval in (33) can be extended to the maximal interval with $T_{C,K,r}=t_0(\epsilon)=\epsilon^{-3}\tau_0(\epsilon)$. Bound (13) of Theorem 1 is proved due to the estimate (38) and the coercivity bound (31), while the definition of C may need a minor adjustment.

4. **Proof of Theorem 2.** Here we use the formalism of Section 2 and prove Theorem 2.

We consider global solutions to the generalized KdV equations with $p\geq 4$. It follows from [22] that for any global solution $W\in C(\mathbb{R},H^s(\mathbb{R}))$ with $s\geq 6$, there exists positive constants A and K such that

$$\delta(\tau):=\sup_{\tau'\in[-\tau,\tau]}\|W(\cdot,\tau')\|_{H^s}\leq A(1+|\tau|^{s-1})\leq Ae^{K|\tau|}\quad\text{for every}\quad\tau\in\mathbb{R}. \tag{40}$$

Again, we neglect mentioning that all constants depend on the choice of $s\geq 6$. We note that the global solution to the generalized KdV equation (5) with $p\geq 5$ exists if the $H^{s_p}(\mathbb{R})$ norm of the initial value W at $\tau=0$ is small, where s_p is given by (11) [13].

For any initial data $(u_{\text{in}},\dot{u}_{\text{in}})\in\ell^2(\mathbb{R})$ satisfying the bound (14), there exists a local solution $(u,\dot{u})\in C^1((-t_0,t_0),\ell^2(\mathbb{Z}))$ to the FPU equation (3), or equivalently, a local solution $(u,q)\in C^1((-t_0,t_0),\ell^2(\mathbb{Z}))$ to the FPU lattice equations (16). The solution can be decomposed according to equation (18).

Let us set $\mathcal{S}:=\mathcal{E}^{1/2}$, where $\mathcal{E}(t)$ is the energy-type quantity defined by (29). The initial bound (14) ensures that $\mathcal{S}(0)\leq C_0\epsilon^{3/2}$ for some constant $C_0>0$, if ϵ_0 is chosen by (30). For fixed ϵ-independent constants $r\in\left(0,\frac{1}{2}\right)$, $C>C_0$, and $K>0$, where K is the same as in the bound (40), let us define the maximal continuation time

$$\begin{aligned}T_{C,K,r}:=\quad&\sup\left\{T_0\in\left(0,(2pK)^{-1}\epsilon^{-3}\log\left(r|\log(\epsilon)|\right)\right]:\right.\\&\left.\qquad\mathcal{S}(t)\leq C\epsilon^{3/2-r},\ t\in[-T_0,T_0]\right\}.\end{aligned} \tag{41}$$

The corresponding evolution time of the generalized KdV equation (5) is given by

$$\tau_0(\epsilon) = (2pK)^{-1} \log\left(r|\log(\epsilon)|\right).$$

Note that $\tau_0(\epsilon)$ here is chosen differently compared to $\tau_0(\epsilon)$ in the proof of Theorem 1. The energy estimate of Lemma 3 is rewritten for $\mathcal{S}$ by

$$\begin{aligned} \left|\frac{d\mathcal{S}}{dt}\right| \quad &\leq \quad C_1(\delta(\tau) + \delta^{2p-1}(\tau))\epsilon^{9/2} \\ &\quad +\epsilon^3 C_2\left[(\delta^{p-1}(\tau) + \delta^{2p-2}(\tau)) + \epsilon^{-1}(\delta^{p-2}(\tau) + \mathcal{S}^{p-2})\mathcal{S}\right]\mathcal{S}, \end{aligned} \tag{42}$$

where C_1 and C_2 are positive constants that are independent of δ and ϵ. Since δ depends on time τ according to the bound (40), we may choose the ϵ-independent positive constant K in the bound (40) sufficiently large such that

$$\begin{aligned} C_2\left[(\delta^{p-1}(\tau) + \delta^{2p-2}(\tau)) + \epsilon^{-1}(\delta^{p-2}(\tau) + C^{p-2}\epsilon^{(3/2-r)(p-2)})C\epsilon^{3/2-r}\right] \\ \leq 2pKe^{2pK|\tau|}, \end{aligned} \tag{43}$$

as long as $\mathcal{S}(t) \leq C\epsilon^{3/2-r}$, since $r \in \left(0, \frac{1}{2}\right)$. Using (42) and (43), we obtain

$$\left|\frac{d}{dt}e^{-e^{2\epsilon^3 pKt}}\mathcal{S}\right| \leq B\epsilon^{9/2}e^{\epsilon^3(2p-1)Kt}e^{-e^{2\epsilon^3 pKt}}, \tag{44}$$

where $B > 0$ is another ϵ-independent constant and the inequality (44) is set for $t > 0$ for simplicity. The estimate for $t < 0$ are similar. By Gronwall's inequality, we obtain

$$\mathcal{S}(t) \leq \left(e^{-1}\mathcal{S}(0) + BF_K\epsilon^{3/2}\right) e^{e^{2pK\tau_0(\epsilon)}}, \quad t \in [-T_{C,K,r}, T_{C,K,r}], \tag{45}$$

where the exponent is extended to the maximal continuation time $t_0(\epsilon) := \epsilon^{-3}\tau_0(\epsilon)$ and the positive ϵ-independent constant F_K is defined by

$$F_K := \int_0^\infty e^{(2p-1)K\tau}e^{-e^{2pK\tau}}d\tau < \infty.$$

From the definition of $\tau_0(\epsilon)$, we have

$$\tau_0(\epsilon) = (2pK)^{-1}\log\left(r|\log(\epsilon)|\right) \quad \Rightarrow \quad e^{e^{2pK\tau_0(\epsilon)}} = \epsilon^{-r}.$$

Thus, we get from (45),

$$\mathcal{S}(t) \leq (C_0 + BF_K)\,\epsilon^{3/2-r}, \quad t \in [-T_{C,K,r}, T_{C,K,r}]. \tag{46}$$

Note that $F_K \to 0$ as $K \to \infty$. One can choose an ϵ-independent constant $C > C_0$ sufficiently large such that

$$C_0 + BF_K \leq C. \tag{47}$$

Under the constraints (40), (43) and (47) on $C > C_0$ and $K > 0$, we can extend the time interval in (41) to the maximal interval with $T_{C,K,r} = t_0(\epsilon) = \epsilon^{-3}\tau_0(\epsilon)$. Bound (15) of Theorem 2 is proved due to the estimate (46) and the coercivity bound (31), while the definition of C may need a minor adjustment.

REFERENCES

[1] D. Bambusi and T. Penati, Continuous approximation of breathers in 1D and 2D DNLS lattices, *Nonlinearity*, **23** (2010), 143–157.

[2] D. Bambusi and A. Ponno, On metastability in FPU, *Comm. Math. Phys.,* **264** (2006), 539–561.

[3] G. N. Benes, A. Hoffman and C. E. Wayne, Asymptotic stability of the Toda m-soliton, *J. Math. Anal. Appl.*, **386** (2012), 445–460.

[4] B. Bidegaray–Fesquet, E. Dumas and G. James, From Newton's cradle to the discrete p-Schrödinger equation, *SIAM J. Math. Anal.*, **45** (2013), 3404–3430.

[5] J. Bona, Y. Liu and N. V. Ngueyn, Stability fo solitary waves in higher-order Sobolev spaces, *Comm Math. Sci.,* **2** (2004), 35–52.

[6] E. Dumas and D. E. Pelinovsky, Justification of the log-KdV equation in granular chains: The case of precompression, *SIAM J. Math. Anal.*, **46** (2014), 4075–4103.

[7] G. Friesecke and R. L. Pego, Solitary waves on FPU lattices, *Nonlinearity,* **12** (1999), 1601–1627; **15** (2002), 1343–1359; **17** (2004), 207–227; **17** (2004), 229–251.

[8] J. Gaison, S. Moskow, J. D. Wright and Q. Zhang, Approximation of polyatomic FPU lattices by KdV equations, *Multiscale Model. Simul.*, **12** (2014), 953–995.

[9] A. Hoffman and C. E. Wayne, Asymptotic two-soliton solutions in the Fermi-Pasta-Ulam model, *J. Dynam. Differential Equations*, **21** (2009), 343–351.

[10] T. Kato, On the Korteweg-de Vries equation, *Manuscript Math.*, **28** (1979), 89–99.

[11] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, *Stud. Appl. Math.,* **8** (1983), 93–128.

[12] C. Kenig, G. Ponce and L. Vega, Well-posedness of the initial-value problem for the Korteweg–De Vries equation, *J. Americ. Math. Soc.*, **4** (1991), 323–347.

[13] C. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg–De Vries equation via the contraction principle, *Comm. Pure Appl. Math.*, **46** (1993), 527–620.

[14] D. Lannes and J. Rauch, Validity of nonlinear geometric optics with times growing logarithmically, *Proc. AMS*, **129** (2000), 1087–1096.

[15] R. M. Miura, C. S. Gardner and M. D. Kruskal, Korteweg–de Vries equations and generalization. II. Existence of conservation laws and constants of motion, *J. Math. Phys.*, **9** (1968), 1204–1209.

[16] T. Mizumachi, Asymptotic stability of lattice solitons in the energy space, *Commun. Math. Phys.*, **288** (2009), 125–144.

[17] T. Mizumachi, Asymptotic stability of N-solitary waves of the FPU lattices, *Archive for Rational Mechanics and Analysis*, **207** (2013), 393–457.

[18] R. L. Pego and M. I. Weinstein, Eigenvalues, and instabilities of solitary waves, *Philos. Trans. Roy. Soc. London A*, **340** (1992), 47–94.

[19] R. L. Pego and M. I. Weinstein, Asymptotic stability of solitary waves, *Comm. Math. Phys.*, **164** (1994), 305–349.

[20] D. Pelinovsky, T. Penati and S. Paleari, Approximation of small-amplitude weakly coupled oscillators with discrete nonlinear Schrödinger equations, *Rev. Math. Phys.*, **28** (2016), 1650015 (25 pages).

[21] G. Schneider and C. E. Wayne, Counter-propagating waves on fluid surfaces and the continuum limit of the Fermi-Pasta-Ulam model, In *International Conference on Differential Equations (Berlin, 1999), vol. 1 (eds B Fiedler, K Gröger, J Sprekels)*, pp. 390–404 (World Sci. Publishing, River Edge, NJ, USA, 2000).

[22] G. Staffilani, On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations, *Duke Math. J.*, **86** (1997), 109–142.

Received February 2016; revised November 2016.

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