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Bifurcations of unstable eigenvalues for Stokes waves derived from conserved energy

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In memory of Vladimir Zakharov, our friend and mentor, whose vision will continue to inspire us.

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ABSTRACT

We address Euler's equations for irrotational gravity waves in an infinitely deep fluid rewritten in conformal variables. Stokes waves are traveling waves with the smooth periodic profiles. In agreement with the previous numerical results, we give a rigorous proof that the zero eigenvalue bifurcation in the linearized equations of motion for co-periodic perturbations occurs at each extremal point of the energy function versus the steepness parameter, provided that the wave speed is not extremal at the same steepness. We derive the leading order of the unstable eigenvalues and, assisted with numerical approximation of its coefficients, we show that the new unstable eigenvalues emerge only in the direction of increasing steepness.

1. Introduction

Ocean swell can be viewed in many cases as a train of almost periodic traveling waves propagating along a fixed direction. Understanding stability properties of periodic wave trains is central to wave forecasting. Such periodic traveling waves were originally found by Stokes [1,2], and hence they are often referenced as the Stokes waves. The existence of Stokes waves with smooth profiles was proven in [3–5]. The existence of the limiting wave with the peaked profile was proven in [6–8].

The stability of Stokes waves is studied either with respect to perturbations co-periodic with the underlying wave (superharmonic), or in a wider space of perturbations periodic with longer periods (subharmonic). In the latter case, the modulational instability, also known as the Benjamin-Feir instability [9,10], is recovered. The modulational instability of Stokes waves was studied rigorously in [11–15]. The high-frequency instabilities discovered in [16] and theoretically studied in [17,18] have the same modulational nature.

Whether subharmonic or superharmonic, the stability properties of traveling waves can be efficiently studied in the limit of small amplitude [19] where the small-amplitude expansions of the Stokes waves offer accurate approximations. For Stokes waves of high steepness and for the limiting wave (with a $2\pi/3$ crest angle), the series expansion diverges and numerical methods must be used instead. Numerical solution of the eigenvalue problem for stability of Stokes waves on a surface of an infinitely deep fluid goes back to [20–22]. In [23,24] the

stability problem is treated as an eigenvalue problem for a large matrix in Fourier basis and is restricted to superharmonic perturbations. Recently, it was realized that the stability spectrum can be determined more efficiently via matrix-free methods [25,26] allowing to extend the stability analysis to nearly limiting Stokes waves [27], and include the Bloch-Floquet theory to cover subharmonic perturbations [28,29].

Fig. 1 presents a schematic dependence of the energy \mathcal{E} (green) and the speed c (red) of the traveling periodic wave continued with respect to the steepness parameter s [24–29], see also [20–22] for the early numerical results suggesting the same behavior of energy and speed versus the steepness. The family of traveling periodic waves bifurcates from the small-amplitude limit $(\mathcal{E}_0 = 0, c_0)$ at s = 0 and oscillates towards the point $(\mathcal{E}_{\lim}, c_{\lim})$, which corresponds to the limiting wave with the peaked profile for the limiting steepness s_{lim} , see [6–8]. A striking aspect of this figure is that every extremal point of the energy corresponds to an instability bifurcation in the co-periodic stability problem, indicated by the presence of a zero eigenvalue with higher algebraic multiplicity compared to the multiplicity imposed by the symmetries of the water wave equations. When the steepness of the periodic wave is increased past the extremal point of the energy, a new pair of real eigenvalues bifurcates in the spectrum of the coperiodic stability problem. It is conjectured in [29] that the family of traveling periodic waves displays infinitely many oscillations with infinitely many instability bifurcations before reaching the limiting peaked wave.

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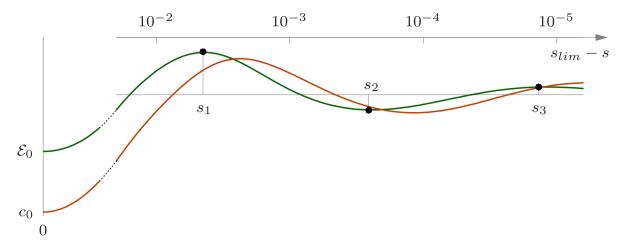


Fig. 1. Oscillations of the energy (green) and the speed (red) as the limiting wave is approached. The figure illustrates $\mathcal{E}_{\text{lim}} - \mathcal{E}$ and $c_{\text{lim}} - c$ as a function of $s_{\text{lim}} - s$, where s_{lim} , \mathcal{E}_{lim} , and c_{lim} represent the steepness, the energy, and the speed of the limiting wave with the peaked profile. The black circles mark the extreme points of the energy, where the instability bifurcation occurs.

The main purpose of this paper is to give a rigorous proof that the instability bifurcation occurs exactly at each extremal point of the energy. Moreover, we show that the extremal points of the energy coincide with those of the horizontal momentum. If the zero eigenvalue has generally geometric multiplicity two and algebraic multiplicity four due to symmetries of the water wave equations, we show that the zero eigenvalue has geometric multiplicity two and algebraic multiplicity of at least six at the instability bifurcation point. This result is given by Theorem 1. In addition, we compute the leading order of the unstable eigenvalues in the co-periodic stability problem and, assuming nonzero coefficients of the leading order, we prove that the new unstable eigenvalues emerge only in the direction of increasing steepness. This result is given by Theorem 2.

For the technical parts of the proofs, we adopt conformal variables for the two-dimensional fluid dynamics developed in [30-34] and used in [23–26] for spectrally accurate numerical approach to the stability problem. The conformal variables allow us to write the problem of finding Stokes wave as a pseudo-differential nonlinear equation [33,35,36], and formulate the stability problem as a matrix-free pseudo-differential eigenvalue problem with periodic coefficients [25,26]. The conserved quantities of the water wave equations [37] are rewritten in conformal variables and impose constraints on solutions of the co-periodic stability problem. Computations of the Jordan blocks and Puiseux expansions for multiple eigenvalues are performed in compliance with the constraints, which act as the Fredholm solvability conditions for solutions at each order of the perturbation theory. Since justification of the Puiseux expansions is fairly known for linear eigenvalue problems [38], we will focus on actual computations rather than on the justification analysis.

The main result of Theorem 1 has been well understood in the dynamics of fluids, based on the numerical results [20,21] and the formal analytical computations [39,40]. Compared to these earlier works which were based on Zakharov's equations of motion [10], we develop the analysis of equations of motion in conformal variables by exploring Babenko's pseudo-differential equation [33] and its linearization. We also go beyond the criterion for the instability bifurcation and compute the leading order of the unstable eigenvalues. Using much more elaborate numerical computations, we confirm that the coefficients of the leading order satisfy the assumptions of our theory.

The paper is organized as follows. Equations of motion in physical and conformal variables are formulated in Section 2 as well as the conserved quantities and the description of the existence and stability problems for the Stokes waves. Section 3 presents the main result on the co-periodic instability bifurcation (Theorem 1). Section 4 presents

the leading order of the unstable eigenvalues (Theorem 2). Section 5 contains numerical approximations of eigenfunctions and eigenvalues at the instability bifurcation to confirm the main prediction that every instability bifurcation generates a new unstable eigenvalue in the direction of the increasing steepness. Further questions are discussed in Section 6.

2. Equations of motion in conformal variables

Let $y=\eta(x,t)$ be the profile for the free surface of an incompressible and irrotational deep fluid in the 2π -periodic domain $\mathbb T$ and in time $t\in\mathbb R$. For a proper definition of the free surface, we add the zero-mean constraint $\int_{\mathbb T} \eta(x,t) dx=0$, which is invariant in the time evolution of Euler's equations with a free surface.

Let $\varphi(x,y,t)$ be the velocity potential, which satisfies the Laplace equation in the time-dependent spatial domain

$$\mathcal{D}(t) := \left\{ (x, y) \in \mathbb{R}^2 : \quad x \in \mathbb{T}, \quad -\infty < y \le \eta(x, t) \right\}$$

subject to the periodic boundary conditions on \mathbb{T} and the decay condition as $y \to -\infty$. The Euler's equations are completed by two additional (kinematic and dynamic) conditions at the free surface $y = \eta(x, t)$:

$$\begin{cases}
 \eta_t + \varphi_x \eta_x - \varphi_y = 0, \\
 \varphi_t + \frac{1}{2} (\varphi_x)^2 + \frac{1}{2} (\varphi_y)^2 + \eta = 0,
 \end{cases}$$
 at $y = \eta(x, t)$, (1)

where the gravity constant g is set to unity for convenience.

Consider now a holomorphic function $z(u,t)=\xi(u,t)+i\eta(u,t)$, which realizes a conformal mapping of the vertical strip in the lower complex half-plane $u\in\mathbb{T}\times i(-\infty,0]$ to the fluid domain $z(\cdot,t)\in\mathcal{D}(t)$ beneath the free surface. The top boundary $\mathrm{Im}\,u=0$ gives the free surface in parametric form $x=\xi(u,t)$ and $y=\eta(u,t)$ written in variables $u\in\mathbb{T}$ and $t\in\mathbb{R}$ with $\xi=u-H\eta$, where H is the periodic Hilbert transform in $L^2(\mathbb{T})$ normalized by the Fourier symbol

$$\hat{\mathcal{H}}_n = \left\{ \begin{array}{ll} i \, \mathrm{sgn}(n), & n \in \mathbb{Z} \backslash \{0\}, \\ 0, & n = 0. \end{array} \right.$$

We also define a positive self-adjoint operator $\mathcal{K} = -\mathcal{H}\partial_u = |\partial_u|$ in $L^2(\mathbb{T})$ with the domain $H^1_{\text{per}}(\mathbb{T})$ and the Fourier symbol

$$\hat{\mathcal{K}}_n = |n|, \quad n \in \mathbb{Z}.$$

It follows from $\xi = u - \mathcal{H}\eta$ that

$$\xi_u = 1 + \mathcal{K}\eta$$
 and $\xi_t = -\mathcal{H}\eta_t$.

The mean value of η in variable $u \in \mathbb{T}$ might be a function of time $t \in \mathbb{R}$ but plays no role in the equations of motion.

By using the constrained Lagrange minimization, see [34] and [36, Appendix A], the system of Euler's equations in physical coordinates (1) can be rewritten as the following system of pseudo-differential equations for the surface velocity potential $\psi(u,t) = \varphi(\xi(u,t),\eta(u,t),t)$, and the free surface elevation $\eta(u,t)$ defined at the top boundary Im u=0.

$$\begin{cases} & \eta_t(1+\mathcal{K}\eta)+\eta_u\mathcal{H}\eta_t+\mathcal{H}\psi_u=0,\\ & \psi_t\eta_u-\psi_u\eta_t+\eta\eta_u+\mathcal{H}\left((1+\mathcal{K}\eta)\psi_t+\psi_u\mathcal{H}\eta_t+\eta(1+\mathcal{K}\eta)\right)=0. \end{cases}$$

The system (2) is the starting point of our work. In the rest of this section, we review the conserved quantities, the traveling wave formulation, the existence problem for traveling waves, and the linear stability problem for traveling waves with respect to co-periodic perturbations.

2.1. Conserved quantities

Taking the mean value of the two equations in system (2) yields the existence of the following two conserved quantities:

$$M(\eta) = \int_{\mathbb{T}} \eta(1 + \mathcal{K}\eta) du,\tag{3}$$

$$P(\psi, \eta) = -\int_{\infty} \psi \eta_u du. \tag{4}$$

Due to the zero-mean constraint $\int_{\mathbb{T}} \eta(x,t) dx = 0$ on the surface elevation η and the chain rule $dx = (1 + \mathcal{K}\eta) du$, we get the constraint $M(\eta) = 0$, or explicitly

$$\int_{\mathbb{T}} \eta(1 + \mathcal{K}\eta) du = 0. \tag{5}$$

With the constraint $M(\eta) = 0$, another conserved quantity follows from the second equation in system (2):

$$Q(\psi, \eta) = \int_{\mathbb{T}} \psi(1 + \mathcal{K}\eta) du, \tag{6}$$

which corresponds to the conserved mean value of the potential ψ on the surface in physical variable $x \in \mathbb{T}$ due to the chain rule $dx = (1 + \mathcal{K}\eta)du$.

The conserved quantities (3), (4), and (6) follow from the general study of symmetries and conserved quantities for Euler's equations in physical coordinates in [37], where $M(\eta)$, $P(\psi,\eta)$, and $Q(\psi,\eta)$ are referred to as *mass*, the horizontal momentum, and the vertical momentum. The same list of conserved quantities in the conformal variable $u \in \mathbb{T}$ can also be found in [34]. The two components of momentum can be expressed in the complex form

$$Q - iP = \int_{\mathbb{T}} \psi z_u \, du,\tag{7}$$

where $z_u = \xi_u + i\eta_u = 1 + \mathcal{K}\eta + i\eta_u$.

To derive the energy conservation, we use the zero-mean constraint (5) and the conservation of $Q(\psi,\eta)$ in (6). Applying $\mathcal H$ to the second equation of system (2) with $\mathcal H^2=-\mathrm{Id}$ in the space of 2π -periodic function with zero mean, we obtain

$$\psi_t(1 + \mathcal{K}\eta) + \psi_u \mathcal{H}\eta_t + \eta(1 + \mathcal{K}\eta) - \mathcal{H}(\psi_t \eta_u - \psi_u \eta_t + \eta \eta_u) = 0.$$
 (8)

Multiplying the first equation of system (2) by ψ_t and Eq. (8) by η_t , integrating over the period of \mathbb{T} , and subtracting one equation from another, we integrate by parts and obtain the conserved *energy* (Hamiltonian) in the form:

$$H(\psi, \eta) = \frac{1}{2} \int_{\mathbb{T}} \left(\psi \mathcal{K} \psi + \eta^2 (1 + \mathcal{K} \eta) \right) du. \tag{9}$$

The energy $H(\psi, \eta)$ is the main quantity in the stability analysis of the traveling waves.

2.2. Formulation of equations of motion in the traveling frame

Let us write the first equation of system (2) and Eq. (8) in the reference frame moving with the wave speed c:

$$\begin{cases} & \eta_t(1+\mathcal{K}\eta)+\eta_u\mathcal{H}\eta_t+\mathcal{H}\psi_u-c\eta_u=0, \\ & \psi_t(1+\mathcal{K}\eta)+\psi_u\mathcal{H}\eta_t+\eta(1+\mathcal{K}\eta)-c\psi_u-\mathcal{H}(\psi_t\eta_u-\psi_u\eta_t+\eta\eta_u)=0, \end{cases}$$

where u now stands for u - ct. Let us introduce the following change of variables by

$$\psi = -c\mathcal{H}\eta + \zeta,\tag{10}$$

after which the equations of motion yield,

$$\begin{cases} \eta_t(1+\mathcal{K}\eta) + \eta_u \mathcal{H}\eta_t + \mathcal{H}\zeta_u = 0, \\ \zeta_t(1+\mathcal{K}\eta) + \zeta_u \mathcal{H}\eta_t + \eta(1+\mathcal{K}\eta) - c\zeta_u - c\mathcal{H}\eta_t - c^2\mathcal{K}\eta \\ -\mathcal{H}\left(\zeta_t\eta_u - \zeta_u\eta_t + \eta\eta_u - c\eta_u\mathcal{H}\eta_t - c\eta_t\mathcal{K}\eta\right) = 0. \end{cases}$$

Substituting $\zeta_u = \mathcal{H}(\eta_t(1 + \mathcal{K}\eta) + \eta_u \mathcal{H}\eta_t)$ from the first equation to the second equation transforms the system of evolution equations to the final form:

$$\begin{cases}
\eta_{t}(1+\mathcal{K}\eta) + \eta_{u}\mathcal{H}\eta_{t} = \mathcal{K}\zeta, \\
\zeta_{t}(1+\mathcal{K}\eta) + \zeta_{u}\mathcal{H}\eta_{t} - \mathcal{H}(\zeta_{t}\eta_{u} - \zeta_{u}\eta_{t}) - 2c\mathcal{H}\eta_{t} = \left(c^{2}\mathcal{K} - 1\right)\eta - \eta\mathcal{K}\eta - \frac{1}{2}\mathcal{K}\eta^{2}.
\end{cases}$$
(11)

We are now ready to set up the existence and linear stability problems for traveling waves.

2.3. Existence of traveling waves

Traveling waves correspond to the reduction $\zeta=0$ for the time-independent solutions of system (11). This gives the scalar pseudo-differential Babenko's equation [33] for the profile $\eta=\eta(u)$:

$$(c^2 \mathcal{K} - 1)\eta = \frac{1}{2} \mathcal{K} \eta^2 + \eta \mathcal{K} \eta. \tag{12}$$

This equation can be obtained as the Euler–Lagrange equation for the action functional

$$\Lambda_{c}(\eta) := \frac{1}{2} \langle (c^{2} \mathcal{K} - 1) \eta, \eta \rangle - \frac{1}{2} \langle \mathcal{K} \eta^{2}, \eta \rangle, \quad \eta \in H^{1}_{per}(\mathbb{T}),$$
(13)

where $\langle f,g\rangle:=\frac{1}{2\pi}\int_{\mathbb{T}}\bar{f}(u)g(u)du$ is a standard normalized inner product in $L^2(\mathbb{T})$. We make the following assumption of existence of traveling waves.

Assumption 1. There exists a family of smooth traveling waves with the even profile $\eta \in C^\infty_{\mathrm{per}}(\mathbb{T})$ satisfying the Babenko equation (12) for $c \in (1,c_*)$ with some $c_* > 1$.

Remark 1. The point c=1 is the bifurcation point of the 2π -periodic solutions with the even single-lobe profile from the zero solution of the Babenko equation (12). It is easy to show, see [19,34,36], that $c^2=1+a^2+\mathcal{O}(a^4)$ and $\eta(u)=a\cos(u)+\mathcal{O}(a^2)$, where a is the small amplitude (steepness) of the 2π -periodic solutions.

Remark 2. As suggested in Fig. 1, the profile $\eta \in C^{\infty}_{per}(\mathbb{T})$ is better parameterized by the steepness, s, rather than the speed, c, because the dependence of speed c on s is oscillatory towards the limiting wave. The details of this dependence are not important for the stability analysis as long as the zero eigenvalue bifurcation (at the extremal point of energy) is different from the extremal point of speed, see Assumption 2.

2.4. Linear stability of traveling waves

Expanding system (11) for (η, ζ) near the traveling wave with the profile $(\eta, 0)$ and truncating the system at the linear terms with respect to the co-periodic perturbation (v, w), we obtain the linearized

equations of motion (also derived in [25]):

$$\begin{cases}
\mathcal{M}v_t = \mathcal{K}w, \\
\mathcal{M}^*w_t - 2c\mathcal{H}v_t = \mathcal{L}v,
\end{cases}$$
(14)

where

$$\mathcal{M} := 1 + \mathcal{K}\eta + \eta'\mathcal{H}$$
 and $\mathcal{M}^* := 1 + \mathcal{K}\eta - \mathcal{H}(\eta' \cdot)$,

and

$$\mathcal{L} := c^2 \mathcal{K} - 1 - \mathcal{K} \eta - \eta \mathcal{K} - \mathcal{K} (\eta \cdot). \tag{15}$$

We note that \mathcal{M}^* is the adjoint operator to a bounded operator \mathcal{M} in $L^2(\mathbb{T})$ with respect to $\langle \cdot, \cdot \rangle$ and that \mathcal{L} is a self-adjoint unbounded operator in $L^2(\mathbb{T})$ with $\mathrm{Dom}(\mathcal{L}) = H^1_{\mathrm{per}}(\mathbb{T})$. Furthermore, \mathcal{L} is the linearized operator of the Babenko equation (12). Also recall that \mathcal{K} is a self-adjoint unbounded operator in $L^2(\mathbb{T})$ with $\mathrm{Dom}(\mathcal{K}) = H^1_{\mathrm{per}}(\mathbb{T})$. It is clear from the Fourier series that $\mathcal{K}1 = 0$, and $\mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1)$.

Remark 3. It follows from the translational symmetry of the Babenko equation (12) that $\mathcal{L}\eta'=0$ with $\eta'\in H^1_{\mathrm{per}}(\mathbb{T})$ under Assumption 1. We also note that

$$\mathcal{L}1 = -(1 + 2\mathcal{K}\eta),\tag{16}$$

which is useful in our computations.

Separating variables in the linearized system (14) yields the spectral stability problem with respect to co-periodic perturbations,

$$\begin{cases}
\mathcal{K}w = \lambda \mathcal{M}v, \\
\mathcal{L}v = \lambda(\mathcal{M}^*w - 2c\mathcal{H}v),
\end{cases}$$
(17)

where $(v,w)\in H^1_{\mathrm{per}}(\mathbb{T})\times H^1_{\mathrm{per}}(\mathbb{T})$ is an eigenfunction and $\lambda\in\mathbb{C}$ is an eigenvalue.

Remark 4. There exist two linearly independent eigenfunctions in the kernel of the spectral stability problem (17) due to the following two symmetries of the underlying physical system. A spatial translation of the Stokes wave results in another solution of the Babenko equation (12), and is associated with a one-dimensional subspace spanned by the eigenfunction $(v, w) = (\eta', 0)$. Similarly, the fluid potential admits gauge transformation $\psi(u, t) \to \psi(u, t) + \psi_0(t)$ for any function $\psi_0(t)$. This property is associated with a one-dimensional subspace spanned by the eigenfunction (v, w) = (0, 1).

3. Criterion for instability bifurcation

We rewrite the spectral stability problem (17) in the matrix form

$$\begin{pmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda \begin{pmatrix} \mathcal{M} & 0 \\ -2c\mathcal{H} & \mathcal{M}^* \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \tag{18}$$

which is rewritten as the generalized eigenvalue problem of the form $A\vec{x} = \lambda B\vec{x}$ with

$$A: H^1_{\mathrm{per}}(\mathbb{T}) \times H^1_{\mathrm{per}}(\mathbb{T}) \to L^2(\mathbb{T}) \times L^2(\mathbb{T}),$$

$$B: L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T}) \times L^2(\mathbb{T}),$$

given by

$$A = \begin{pmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \mathcal{M} & 0 \\ -2c\mathcal{H} & \mathcal{M}^* \end{pmatrix},$$

and $\vec{x} = (v, w) \in H^1_{\text{per}}(\mathbb{T}) \times H^1_{\text{per}}(\mathbb{T})$. Since the embedding of $H^1_{\text{per}}(\mathbb{T})$ into $L^2(\mathbb{T})$ is compact, the resolvent operator $(A - \lambda B)^{-1}$ is compact, and the spectrum of the spectral stability problem (18) consists of eigenvalues with finite algebraic multiplicity.

Remark 5. The bounded operator $\mathcal{M}: L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is invertible with the explicit formula for the inverse operator, see [25, Eq. (13)]. Hence, the bounded operator $B: L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T}) \times L^2(\mathbb{T})$ is also invertible so that the generalized eigenvalue problem $A\vec{x} = \lambda B\vec{x}$ in (18) can be rewritten as the linear eigenvalue problem $B^{-1}A\vec{x} = \lambda \vec{x}$.

The geometric multiplicity of $\lambda=0$ is defined by the dimension of Ker(A). The algebraic multiplicity of $\lambda=0$ is defined by the length of the Jordan chain of generalized eigenvectors

$$\begin{split} A\vec{x}_0 &= 0,\\ A\vec{x}_1 &= B\vec{x}_0,\\ A\vec{x}_2 &= B\vec{x}_1,\\ &\vdots \end{split}$$

as long as $\vec{x}_0, \vec{x}_1, \vec{x}_2, ... \in H^1_{per}(\mathbb{T}) \times H^1_{per}(\mathbb{T})$. In what follows, we compute the Jordan chain of $\lambda = 0$ for the particular operators A and B in (18).

3.1. Eigenvectors of (18) for $\lambda = 0$ due to the symmetries

Since $\mathcal{K}1 = 0$ and $\mathcal{L}\eta' = 0$, the null space of the unbounded operator $A: H^1_{\text{per}}(\mathbb{T}) \times H^1_{\text{per}}(\mathbb{T}) \to L^2(\mathbb{T}) \times L^2(\mathbb{T})$ is at least two-dimensional with

$$\begin{pmatrix} v \\ w \end{pmatrix} = a_1 \begin{pmatrix} \eta' \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{19}$$

where $(a_1, a_2) \in \mathbb{R}^2$. Due to the Hamiltonian symmetry, see (21) and (22) below, the generalized null space of the spectral stability problem (18) is at least four-dimensional with at least two generalized eigenfunctions.

Definition 1. We say that the periodic wave with the profile $\eta \in C^{\infty}_{per}(\mathbb{T})$ is at the stability threshold if the generalized null space of the spectral stability problem (18) has algebraic multiplicity exceeding four

Hypothetically, the stability threshold of Definition 1 can be hit if either the null space of A becomes at least three-dimensional or the null space of A remains two-dimensional but the generalized null space of $B^{-1}A$ becomes at least six-dimensional. Since $Ker(\mathcal{K}) = span(1)$, the first possibility could only be realized if \mathcal{L} has a double zero eigenvalue, which is indeed observed numerically in [25]. However, it was realized in [21,39] (see also [29]) that the points of the double zero eigenvalue of \mathcal{L} correspond to the fold points in the dependence of speed c versus steepness s, see the red curve in Fig. 1. In this case, the family of solutions of the Babenko equation (12) fails to continue with respect to parameter c, but the generalized null space of the spectral stability problem (18) still has the algebraic multiplicity equals to four. Therefore, the stability threshold of Definition 1 can only be realized away from the fold points, which are eliminated by the following assumption.

Assumption 2. For the given value of $c \in (1, c_*)$, we have $Ker(\mathcal{L}) = span(\eta')$.

Remark 6. It follows from Assumption 2 that the mapping $c \mapsto \eta \in C^{\infty}_{per}(\mathbb{T})$ is smooth, since the profile of η is even and the kernel of \mathcal{L} is spanned by the odd eigenfunction η' . Hence, we can differentiate the Babenko equation (12) in c and obtain

$$\mathcal{L}\partial_c \eta + 2cK\eta = 0, \quad \Rightarrow \quad \partial_c \eta = -2c\mathcal{L}^{-1}K\eta, \tag{20}$$

where $\partial_c \eta \in C^{\infty}_{per}(\mathbb{T})$ and \mathcal{L}^{-1} is uniquely defined on the subspace of even functions in $L^2(\mathbb{T})$.

Under Assumption 2, the periodic solutions to $A\vec{x}_0=0$ are spanned by (19). The first element of the Jordan chain $A\vec{x}_1=B\vec{x}_0$ is defined by the periodic solutions $(v_1,w_1)\in H^1_{\rm per}(\mathbb{T})\times H^1_{\rm per}(\mathbb{T})$ of the linear inhomogeneous equations:

$$\begin{cases}
\mathcal{K}w_1 = a_1 \mathcal{M}\eta', \\
\mathcal{L}v_1 = -2ca_1 \mathcal{H}\eta' + a_2 \mathcal{M}^*1.
\end{cases}$$
(21)

Since $\mathcal{M}\eta'=\eta'$ and $\langle 1,\eta'\rangle=0$, there exists $w_1\in \mathrm{Dom}(\mathcal{K})$ from the first equation in the system (21). For unique definition of w_1 , we take projection of w_1 to 1 to be zero, after which we get $w_1=a_1\mathcal{H}\eta$. Since η is even, both $\mathcal{H}\eta$ and η' are odd. Hence, η' has opposite parity compared to $\mathcal{H}\eta'$ and $\mathcal{M}^*1=1+2\mathcal{K}\eta$ so that there exists $v_1\in \mathrm{Dom}(\mathcal{L})$ from the second equation in the system (21). For unique definition of v_1 , we take projection of v_1 to η' to be zero. By using (16) and (20), we get the explicit solution $v_1=-a_1\partial_c\eta-a_2$, where $\partial_c\eta$ is even and $\mathcal{H}\eta$ is odd. Thus, the periodic solutions to the first element of the Jordan chain $A\vec{x}_1=B\vec{x}_0$ are spanned by

$$\begin{pmatrix} v \\ w \end{pmatrix} = a_1 \begin{pmatrix} -\partial_c \eta \\ \mathcal{H} \eta \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \tag{22}$$

so that the generalized null space of the spectral stability problem (18) is at least four-dimensional with at least two generalized eigenfunctions.

3.2. Eigenvectors of (18) for $\lambda = 0$ due to the zero eigenvalue bifurcation

In what follows, we compute the second element of the Jordan chain $A\vec{x}_2 = B\vec{x}_1$ and obtain the criterion for the generalized null space of the spectral stability problem (18) to be at least six-dimensional. To do so, we define the wave momentum $\mathcal{P}(c)$ and the wave energy $\mathcal{E}(c)$ related to the wave profile $\eta \in C^\infty_{\mathrm{per}}(\mathbb{T})$ as

$$\mathcal{P}(c) := P(\psi = -c\mathcal{H}\eta, \eta) = c\langle \mathcal{K}\eta, \eta \rangle \tag{23}$$

and

$$\mathcal{E}(c) := H(\psi = -c\mathcal{H}\eta, \eta) = \frac{c^2}{2} \langle \mathcal{K}\eta, \eta \rangle + \frac{1}{2} \langle \eta^2, (1 + \mathcal{K}\eta) \rangle, \tag{24}$$

where $P(\psi, \eta)$ and $H(\psi, \eta)$ are given by (4) and (9).

The following theorem presents the main result on the criterion for instability bifurcation.

Theorem 1. Under Assumptions 1 and 2, the generalized null space of the spectral problem (18) is at least six-dimensional if and only if $\mathcal{P}'(c) = 0$ or, equivalently, $\mathcal{E}'(c) = 0$.

Proof. The second element of the Jordan chain $A\vec{x}_2 = B\vec{x}_1$ is defined by the periodic solutions $(v_2, w_2) \in H^1_{per}(\mathbb{T}) \times H^1_{per}(\mathbb{T})$ of the linear inhomogeneous equations:

$$\begin{cases}
\mathcal{K}w_2 &= \mathcal{M}v_1, \\
\mathcal{L}v_2 &= -2c\mathcal{H}v_1 + \mathcal{M}^*w_1.
\end{cases}$$
(25)

The system (25) is written explicitly as

$$\begin{cases}
\mathcal{K}w_2 &= -a_1 \mathcal{M} \partial_c \eta - a_2 \mathcal{M} 1, \\
\mathcal{L}v_2 &= 2ca_1 \mathcal{H} \partial_c \eta + a_1 \mathcal{M}^* \mathcal{H} \eta.
\end{cases}$$
(26)

Since

$$\begin{split} \langle 1, \mathcal{M}f \rangle &= \langle \mathcal{M}^*1, f \rangle = \langle (1 + 2\mathcal{K}\eta), f \rangle, \\ \langle \eta', \mathcal{M}^*f \rangle &= \langle \mathcal{M}\eta', f \rangle = \langle \eta', f \rangle, \\ \langle \eta', \mathcal{H}f \rangle &= -\langle \mathcal{H}\eta', f \rangle = \langle \mathcal{K}\eta, f \rangle, \end{split}$$

Fredholm theorem implies that there exist periodic solutions of the linear inhomogeneous system (26) if and only if the following linear homogeneous system on (a_1, a_2) admits a nonzero solution:

$$\begin{pmatrix} -\langle (1+2\mathcal{K}\eta), \partial_c \eta \rangle & -\langle (1+2\mathcal{K}\eta), 1 \rangle \\ \langle \eta', \mathcal{H}\eta \rangle + 2c\langle \mathcal{K}\eta, \partial_c \eta \rangle & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{27}$$

Taking derivative of the constraint (5) with respect to c yields

$$\langle (1 + 2\mathcal{K}\eta), \partial_c \eta \rangle = 0.$$

On the other hand, since K is self-adjoint and K1 = 0, we have

$$\langle 1, \mathcal{K} \eta \rangle = \langle \mathcal{K} 1, \eta \rangle = 0.$$

Hence, the linear system (27) can be rewritten in the equivalent form as

$$\begin{pmatrix} 0 & -1 \\ \langle \eta, \mathcal{K} \eta \rangle + 2c \langle \mathcal{K} \eta, \partial_c \eta \rangle & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $a_2 = 0$, whereas $a_1 \neq 0$ if and only if

$$0 = \langle \mathcal{K}\eta, \eta \rangle + 2c \langle \mathcal{K}\eta, \partial_c \eta \rangle = \frac{d}{dc} c \langle \mathcal{K}\eta, \eta \rangle = \mathcal{P}'(c). \tag{28}$$

In the case of $\mathcal{P}'(c)=0$, the periodic solutions to the second element of the Jordan chain $A\vec{x}_2=B\vec{x}_1$ are represented in the form $(v_2,w_2)=a_1(\tilde{v}_2,\tilde{w}_2)$ for $a_2=0$, where $(\tilde{v}_2,\tilde{w}_2)\in H^1_{\mathrm{per}}(\mathbb{T})\times H^1_{\mathrm{per}}(\mathbb{T})$ are uniquely defined from solutions of the linear inhomogeneous equations

$$\begin{cases}
\mathcal{K}\tilde{w}_{2} &= -\mathcal{M}\partial_{c}\eta, \\
\mathcal{L}\tilde{v}_{2} &= 2c\mathcal{H}\partial_{c}\eta + \mathcal{M}^{*}\mathcal{H}\eta - \frac{p'(c)}{\|p'\|^{2}}\eta',
\end{cases} (29)$$

subject to the orthogonality conditions

$$\langle 1, \tilde{w}_2 \rangle = 0$$
 and $\langle \eta', \tilde{v}_2 \rangle = 0.$ (30)

To prove that the generalized null space of the spectral problem (18) is at least six-dimensional, we consider the third element of the Jordan chain $A\vec{x}_3 = B\vec{x}_2$ defined by the periodic solutions $(v_3, w_3) \in H^1_{\text{ner}}(\mathbb{T}) \times H^1_{\text{ner}}(\mathbb{T})$ of the linear inhomogeneous equations

$$\begin{cases}
\mathcal{K}w_3 = a_1 \mathcal{M} \tilde{v}_2, \\
\mathcal{L}v_3 = -2ca_1 \mathcal{H} \tilde{v}_2 + a_1 \mathcal{M}^* \tilde{w}_2.
\end{cases}$$
(31)

Since η is even and operators \mathcal{K} , \mathcal{M} and \mathcal{L} are parity preserving, we obtain from (29) and the orthogonality conditions (30) that \tilde{w}_2 is even and \tilde{v}_2 is odd. Hence, odd $\mathcal{M}\tilde{v}_2$ is orthogonal to even 1 and even $-2c\mathcal{H}\tilde{v}_2+\mathcal{M}^*\tilde{w}_2$ is orthogonal to odd η' . By Fredholm's theorem, the periodic solutions to the third element of the Jordan chain $A\vec{x}_3=B\vec{x}_2$ are represented in the form $(v_3,w_3)=a_1(\tilde{v}_3,\tilde{w}_3)$, where $(\tilde{v}_3,\tilde{w}_3)\in H^1_{\mathrm{per}}(\mathbb{T})\times H^1_{\mathrm{per}}(\mathbb{T})$ are uniquely defined from solutions of the linear inhomogeneous equations

$$\begin{cases}
\mathcal{K}\tilde{w}_{3} &= \mathcal{M}\tilde{v}_{2}, \\
\mathcal{L}\tilde{v}_{3} &= -2c\mathcal{H}\tilde{v}_{2} + \mathcal{M}^{*}\tilde{w}_{2}.
\end{cases} (32)$$

subject to the orthogonality conditions

$$\langle 1, \tilde{w}_3 \rangle = 0$$
 and $\langle \eta', \tilde{v}_3 \rangle = 0.$ (33)

From the same parity argument and the orthogonality conditions (33), we conclude that $\tilde{w}_3 \in H^1_{per}(\mathbb{T})$ is odd and $\tilde{v}_3 \in H^1_{per}(\mathbb{T})$ is even. Thus, the generalized null space of the spectral problem (18) is at least six-dimensional if and only if $\mathcal{P}'(c) = 0$.

It remains to show that the critical points of $\mathcal{P}(c)$ coincide with the critical points of the energy $\mathcal{E}(c)$. Differentiating the action $\Lambda_c(\eta)$ given by (13) in c yields

$$\frac{d}{dc}\Lambda_c(\eta) = c\langle \mathcal{K}\eta, \eta \rangle + \langle (c^2\mathcal{K}\eta - \eta - \frac{1}{2}\mathcal{K}\eta^2 - \eta\mathcal{K}\eta), \partial_c \eta \rangle = \mathcal{P}(c), \tag{34}$$

where the quantity in the brackets vanishes due to the Babenko equation (12). Since $\Lambda_c(\eta) = c\mathcal{P}(c) - \mathcal{E}(c)$, we obtain

$$\frac{d}{dc}\Lambda_c(\eta) = \mathcal{P}(c) + c\mathcal{P}'(c) - \mathcal{E}'(c),$$

which yields $\mathcal{E}'(c) = c\mathcal{P}'(c)$. \square

Remark 7. We have added the orthogonal projection to the second equation of system (29) even though $\mathcal{P}'(c) = 0$. This is useful for numerical approximations as well as for the derivation of the leading order of the unstable eigenvalues in Theorem 2.

Remark 8. The two orthogonality conditions used in the proof of Theorem 1 can be stated for every eigenfunction $(v,w)\in H^1_{\operatorname{per}}(\mathbb{T})\times H^1_{\operatorname{per}}(\mathbb{T})$ of the spectral problem (18) with $\lambda\neq 0$. Indeed, the two Fredholm constraints

$$0 = \langle 1, \mathcal{K}w \rangle = \lambda \langle 1, \mathcal{M}v \rangle = \lambda \langle \mathcal{M}^*1, v \rangle,$$

 $0 = \langle \eta', \mathcal{L}v \rangle = \lambda \left(\langle \eta', \mathcal{M}^*w \rangle - 2c \langle \eta', \mathcal{H}v \rangle \right) = \lambda \left(\langle \mathcal{M}\eta', w \rangle + 2c \langle \mathcal{H}\eta', v \rangle \right)$ imply

$$\langle (1+2\mathcal{K}\eta), v \rangle = 0, \qquad \langle \eta', w \rangle - 2c\langle \mathcal{K}\eta, v \rangle = 0. \tag{35}$$

The first orthogonality condition in (35) is a linearization of the constraint (5). The second orthogonality condition in (35) is a linearization of the momentum conservation $P(\psi, \eta)$ with the decomposition (10):

$$P(\psi, \eta) = c \langle \mathcal{K} \eta, \eta \rangle - \langle \eta_u, \zeta \rangle,$$

since (v, w) is the perturbation of the traveling wave with the profile $(\eta, 0)$ in variables (η, ζ) .

Remark 9. It follows from the proof of Theorem 1 that the Jordan canonical form of $B^{-1}A$ at $\lambda = 0$ if $\mathcal{P}'(c) \neq 0$ is

$$\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

If $\mathcal{P}'(c) = 0$ and $B \neq 0$, where the numerical coefficient B is given by (36) below, the Jordan canonical form of $B^{-1}A$ for $\lambda = 0$ is

4. The leading order of unstable eigenvalues

We study the splitting of the multiple zero eigenvalue of the spectral problem (18) for the values of c close to a critical point of $\mathcal{P}(c)$ in Theorem 1. The following theorem gives the main result.

Theorem 2. Under Assumptions 1 and 2, assume that $c_0 \in (1, c_*)$ is an extremal point of $\mathcal{P}(c)$ such that $\mathcal{P}'(c_0) = 0$, $\mathcal{P}''(c_0) \neq 0$, and $\mathcal{B} \neq 0$, where

$$\mathcal{B} := \langle \eta', \tilde{w}_2 \rangle - 2c \langle K \eta, \tilde{v}_2 \rangle \tag{36}$$

is defined from solutions of (32) computed from solutions of (29). Then, there is $\epsilon_0>0$ such that for every $c\in(c_0,c_0+\epsilon_0)$, the spectral stability problem (18) admits two (small) real eigenvalues $\pm\lambda(c)$ with $\lambda(c)>0$ near 0 if $\mathcal{BP''}(c_0)<0$ and two (small) purely imaginary eigenvalues $\pm i\omega(c)$ with $\omega(c)>0$ near 0 if $\mathcal{BP''}(c_0)>0$. The real and purely imaginary eigenvalues are exchanged to the opposite if $c\in(c_0-\epsilon_0,c_0)$.

Proof. Since $\mathcal{P}'(c_0)=0$ and $c\in(c_0-\epsilon_0,c_0+\epsilon_0)$ for small $\epsilon_0>0$, we expand

$$P'(c) = P''(c_0)(c - c_0) + \mathcal{O}((c - c_0)^2)$$
(37)

with $\mathcal{P}''(c_0) \neq 0$ due to our assumption. Let $\epsilon := |c - c_0| \in (0, \epsilon_0)$ be a small parameter. Since only one Jordan block for the double eigenvalue is extended if $\mathcal{P}'(c_0) = 0$, see Remark 9, solutions to the spectral stability problem (18) are found by Puiseux expansion for the double zero eigenvalue:

$$\begin{cases} v = \eta' + \sqrt{\epsilon}v_1 + \epsilon v_2 + \epsilon \sqrt{\epsilon}v_3 + \epsilon^2 v_4 + \mathcal{O}(\epsilon^2 \sqrt{\epsilon}), \\ w = 0 + \sqrt{\epsilon}w_1 + \epsilon w_2 + \epsilon \sqrt{\epsilon}w_3 + \epsilon^2 w_4 + \mathcal{O}(\epsilon^2 \sqrt{\epsilon}), \\ \lambda = 0 + \sqrt{\epsilon}\lambda_1 + \epsilon\lambda_2 + \epsilon \sqrt{\epsilon}\lambda_3 + \epsilon^2\lambda_4 + \mathcal{O}(\epsilon^2 \sqrt{\epsilon}), \end{cases}$$
(38)

where all correction terms are to be found recursively. Since the admissible values of λ_1 are found at the order of $\mathcal{O}(\epsilon^4)$ and the admissible values of λ_2 , λ_3 , etc are found at the higher orders of ϵ , we will not write any correction terms related to λ_2 , λ_3 , etc. They are identical to the correction terms related to λ_1 .

At the order of $\mathcal{O}(\sqrt{\epsilon})$, we obtain the linear inhomogeneous system (21) with $a_1 = \lambda_1$ and $a_2 = 0$, hence the solution is

$$v_1 = -\lambda_1 \partial_c \eta, \qquad w_1 = \lambda_1 \mathcal{H} \eta,$$

in agreement with (22).

At the order of $\mathcal{O}(\epsilon)$, we obtain the linear inhomogeneous system (26) with $a_1 = \lambda_1^2$ and $a_2 = 0$. Recall that the solution of (26) exists if and only if $\mathcal{P}'(c) = 0$, which is not the case if $c \neq c_0$ due to $\epsilon \neq 0$. Therefore, we represent the solution of (26) in the form

$$v_2 = \lambda_1^2 \tilde{v}_2, \qquad w_2 = \lambda_1^2 (\tilde{w}_2 + \alpha),$$

where $(\tilde{v}_2, \tilde{w}_2)$ is a solution of the linear inhomogeneous system (29) uniquely defined under the orthogonality conditions (30) and $\alpha \in \mathbb{R}$ is a parameter to be determined from the orthogonality condition at the order of $\mathcal{O}(\epsilon^2)$ due to $\text{Ker}(\mathcal{K}) = \text{span}(1)$.

At the order of $\mathcal{O}(\epsilon\sqrt{\epsilon})$, we obtain the linear inhomogeneous system,

$$\begin{cases}
\mathcal{K}w_3 = \lambda_1^3 \mathcal{M}\tilde{v}_2, \\
\mathcal{L}v_3 = \lambda_1^3 (-2c\mathcal{H}\tilde{v}_2 + \mathcal{M}^*\tilde{w}_2 + \alpha \mathcal{M}^*1).
\end{cases}$$
(39)

which can be compared with (21) and (31). The solution exists in the form

$$v_3 = \lambda_1^3 (\tilde{v}_2 - \alpha), \qquad w_3 = \lambda_1^3 \tilde{w}_3,$$

where $(\tilde{v}_3, \tilde{w}_3)$ is a solution of the linear inhomogeneous Eq. (32) uniquely defined under orthogonality conditions (33).

At the order of $\mathcal{O}(\epsilon^2)$, we obtain the linear inhomogeneous system,

$$\begin{cases}
\mathcal{K}w_4 &= \lambda_1^4(\mathcal{M}\tilde{v}_3 - \alpha \mathcal{M}1), \\
\mathcal{L}v_4 &= \lambda_1^4(-2c\mathcal{H}\tilde{v}_3 + \mathcal{M}^*\tilde{w}_3) + \lambda_1^2 \epsilon^{-1} \frac{p'(c)}{\|\eta'\|^2} \eta',
\end{cases} (40)$$

where the projection term came from the order $\mathcal{O}(\epsilon)$ in the linear inhomogeneous system (29). Since $\mathrm{Ker}(\mathcal{K})=\mathrm{span}(1)$, the value of $\alpha\in\mathbb{R}$ is uniquely found from the existence of the solution $w_4\in H^1_{\mathrm{per}}(\mathbb{T})$ by the Fredholm theorem:

$$\alpha = \frac{\langle 1, \mathcal{M} \tilde{v}_3 \rangle}{\langle 1, \mathcal{M} 1 \rangle} = \langle (1 + 2 \mathcal{K} \eta), \tilde{v}_3 \rangle,$$

where we have used $\langle 1,\mathcal{M}1\rangle = \langle (1+2\mathcal{K}\eta),1\rangle = 1$. Although α is uniquely defined from the first equation of the system (40), it does not contribute to the second equation of the system (40) and therefore does not change the leading order of the unstable eigenvalues. Since $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$, a solution $v_4 \in H^1_{\mathrm{per}}(\mathbb{T})$ to the second equation of the system (40) exists if and only if

$$\lambda_1^4 \langle \eta', (-2c\mathcal{H}\tilde{v}_3 + \mathcal{M}^* \tilde{w}_3) \rangle + \lambda_1^2 \epsilon^{-1} \mathcal{P}'(c) = 0.$$

By using (36) and (37), we can rewrite the characteristic equation at the leading order as

$$\lambda_1^4 \mathcal{B} + \lambda_1^2 \operatorname{sgn}(c - c_0) \mathcal{P}''(c_0) = 0, \tag{41}$$

where $B \neq 0$ due to our assumption. A nonzero solution for λ_1 exists since $\mathcal{P}''(c_0) \neq 0$ due to our assumption. For $c \in (c_0, c_0 + \epsilon_0)$, we have $\lambda_1^2 > 0$ if $\mathcal{BP}''(c_0) < 0$ and $\lambda_1^2 < 0$ if $\mathcal{BP}''(c_0) > 0$. The sign of λ_1^2 is exchanged to the opposite if $c \in (c_0 - \epsilon_0, c_0)$. This concludes the proof. \square

Remark 10. When $(\tilde{v}_2,\tilde{w}_2)$ is obtained from the system (29) for $c \neq c_0$, we add the additional term $-\frac{\mathcal{P}'(c)}{\|\eta'\|^2}\eta'$ in the system (29) required by the Fredholm theorem. This term of order $\mathcal{O}(\epsilon)$ is compensated for by the term $+\epsilon^{-1}\frac{\mathcal{P}'(c)}{\|\eta'\|^2}\eta'$ of order $\mathcal{O}(\epsilon^2)$ in the system (40). The orders are consistent with the assumption $\mathcal{P}''(c_0) \neq 0$, which ensures that $\mathcal{P}'(c) = \mathcal{O}(\epsilon)$.

Remark 11. As Fig. 1 shows, the periodic wave with the even profile $\eta \in C^{\infty}_{per}(\mathbb{T})$ is continued numerically with respect to the steepness parameter s. Based on the numerical observations in Fig. 1, there is exactly one fold point between each extremal point of $\mathcal{P}(c)$ or, equivalently, $\mathcal{E}(c)$, so that $\frac{dc}{ds}$ alternates its sign at each point, where

Table 1 Parameters of Stokes waves at the first two extrema of energy $\mathcal E$ or, equivalently, the horizontal momentum $\mathcal P$ with the coefficient $\mathcal B$ in (36).

S	c	\mathcal{E}	\mathcal{P}	В
0.13660354990	1.0921379	0.46517718146	0.44729319629	11.01822
0.14079654715	1.0922868	0.45770578965	0.44045605242	10.96232

 $\mathcal{P}'(c_0) = 0$. Similarly, $\mathcal{P}''(c_0)$ alternates its sign between the extremal points. Since we show numerically in Section 5 that the value of \mathcal{B} has the same sign for each instability bifurcation, the new pair of real (unstable) eigenvalues λ bifurcates in the direction of increasing steepness s at each extremal point of $\mathcal{P}(c)$.

5. Numerical approximations

Stokes waves of large steepness are beyond the applicability of the small-amplitude expansions. Table 1 gives the first and second critical points of the energy $\mathcal E$ at steepness s_1 and s_2 , which are also seen in Fig. 1. It is then necessary to use other approximations of Stokes waves with large steepness before the stability problem can be studied. There are two challenging problems that have to be treated numerically, the first one is obtaining a solution of the Babenko equation (12) with high accuracy, and the second one is finding eigenvalues of the stability problem (18). Once the eigenvalues are found, we can compare them with the leading order given by (41) to cross-validate numerics and theory.

5.1. The Babenko equation

We adopt the strategy from [25,27] to find Stokes waves numerically. The entire branch of Stokes waves is found by the continuation method with respect to the speed parameter c. Given a Stokes wave $\eta^{(0)}(u)$ with speed $c^{(0)}$ that solves $S(c^{(0)},\eta^{(0)})=0$, where S denotes the nonlinear Babenko equation (12), we apply Newton's method to find a new solution $(c^{(1)},\eta^{(1)})$. The initial approximation to Stokes wave with $c^{(1)}$ is chosen to be $\eta^{(0)}+\delta\eta$ from the expansion

$$0 = S(c^{(1)}, \eta^{(0)} + \delta \eta) = S(c^{(1)}, \eta^{(0)}) + \mathcal{L}(c^{(1)}, \eta^{(0)}) \delta \eta + \cdots, \tag{42}$$

where the neglected terms are quadratic and higher order in terms of $\delta \eta$ and $\mathcal{L}(c^{(1)}, \eta^{(0)})$ is the linearized Babenko operator computed at the profile $\eta^{(0)}$ for the speed $c^{(1)}$. Once the nonlinear terms in (42) are neglected, the approximate equation for $\delta \eta$ yields

$$\mathcal{L}(c^{(1)}, \eta^{(0)})\delta\eta = -\mathcal{S}(c^{(1)}, \eta^{(0)}),\tag{43}$$

which is solved in the Fourier space by means of the minimum residual method [41] (MINRES). The linearized Babenko's operator $\mathcal L$ is self-adjoint in $L^2(\mathbb T)$, however it is not positive definite. This makes MINRES the preferred method of computing solutions for the correction term (43). Once the linear system in (43) is solved, the approximate solution is updated via $\eta^{(0)} \to \eta^{(0)} + \delta \eta$, and the new $\delta \eta$ is found from (43) with updated $\eta^{(0)}$. This algorithm is repeated until a convergence criterion $\|S(c^{(1)},\eta^{(0)}+\delta \eta)\|_{L^2} \leq 10^{-28}$ is reached, at which step we assign $\eta^{(1)}=\eta^{(0)}+\delta \eta$ for this value of $c^{(1)}$. For variable precision arithmetic, the GNU MPFR [42] and GNU MPC [43] libraries are used, and the fast Fourier Transform (FFT) C library is written based on [44]. The convergence rate of Fourier series is improved by means of auxiliary conformal mapping based on Jacobi elliptic function, see [45] and applications of this method in [27].

5.2. Generalized eigenfunctions

We construct the eigenfunctions in the proof of Theorem 1 using a variant of orthogonal projection algorithm programmed in double precision arithmetic. Afterwards, the chain of generalized eigenfunctions

is found by numerically solving linear inhomogeneous systems with the preconditioned MINRES. A symmetric positive definite preconditioner is defined by the strictly positive operator $(1+c^2\mathcal{K})$ to improve the linear solver convergence rate.

Fig. 2 shows the three generalized eigenfunctions of the Jordan chains (v_1, w_1) , (v_2, w_2) , and (v_3, w_3) defined via the linear inhomogeneous systems (21) with a = 1, $a_2 = 0$, (29), and (31) for the Stokes wave at the first two extrema s_1 and s_2 of the energy in Table 1.

Fig. 3 illustrates the convergence rate of the preconditioned MINRES for the system (29) to find the eigenfunction (v_2, w_2) for Stokes waves at s_1 and s_2 . The number of Fourier modes to represent the Stokes wave and the (generalized) eigenfunctions on a uniform grid is N = 8192 at s_1 , and N = 262144 for s_2 .

5.3. Finding eigenvalues of the stability problem

Eigenvalues of the stability problem can be found from numerically solving the eigenvalue problem (18), or equivalently, the quadratic pencil problem

$$\left[\mathcal{M}^*\mathcal{K}^{-1}\mathcal{M}\lambda^2 - 2c\mathcal{H}\lambda - \mathcal{L}\right]\upsilon = 0,$$

where \mathcal{K}^{-1} is defined under the orthogonality condition $\langle (1+2\mathcal{K}\eta),v\rangle=0$, see (35). The quadratic pencil problem was used in [25] for coperiodic perturbations and in [26,29] for subharmonic perturbations via the Bloch-Floquet theory (see also [46] for the numerical Hill method). A new pair of eigenvalues collide at each extrema of the energy. The collision occurs at the origin in the spectral plane, and the eigenvalues become real as shown in Fig. 4 (right panels). It is convenient to show the square of eigenvalue $\lambda^2(\varepsilon)$ as a function of $\varepsilon=c-c_0$ and compare it to the leading order (41) to cross validate theory and numerics. We do so in Fig. 4 (left panels). The direct computation of eigenvalues is obtained via the shift-and-invert method.

Remark 12. It is interesting to note that the values of the coefficient \mathcal{B} are surprisingly close at both extrema of the energy, see Table 1. The coefficient \mathcal{B} is uniquely defined by (36) computed from the Puiseux expansion (38). A further investigation is needed to check if this behavior is universal for all extremal points of the energy.

6. Conclusion

We summarize the main outcome of this work. We have used conformal variables for the two-dimensional Euler's equation in an infinitely deep fluid and computed the leading order of the unstable eigenvalues arising from the zero eigenvalue bifurcation of the Stokes waves with respect to co-periodic perturbations. The zero eigenvalue bifurcation occurs at every extremal point of the energy or, equivalently, the horizontal momentum. The leading order of the unstable eigenvalues computed numerically shows that the new unstable eigenvalues emerge in the direction of the increasing steepness of the Stokes wave.

This work opens the road to analytic understanding of bifurcations of the unstable spectral bands in the modulational instability of Stokes waves by using the Bloch–Floquet theory. Numerical results have been computed recently in [28,29] and show interesting transformations of the spectral bands when the real unstable eigenvalues bifurcate in the space of anti-periodic and co-periodic perturbations. The figure- ∞ instability arises at the co-periodic instability bifurcation, and this transformation is described by the characteristic equation which extends the leading-order approximation (41) by the parameter of the Bloch–Floquet theory. Details of the derivation of the characteristic equation are currently in progress. The recent work [47] describes a similar transformation to the figure- ∞ instability in the local model of the focusing modified Korteweg–de Vries equation, which has been studied by using integrability of the model.

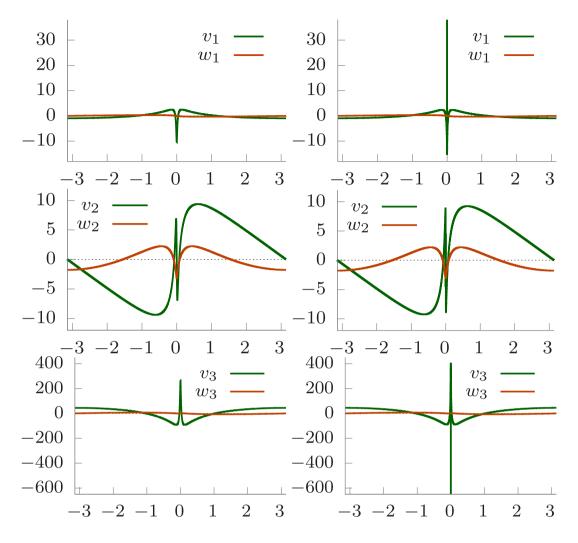


Fig. 2. The generalized eigenvectors (v_1, w_1) , (v_2, w_2) and (v_3, w_3) defined via Eqs. (21) with $a_1 = 1$, $a_2 = 0$, (29) and (31) (top to bottom) for the first two critical points of the energy at $s_1 = 0.13660354990$ (left) and $s_2 = 0.14079654715$ (right).

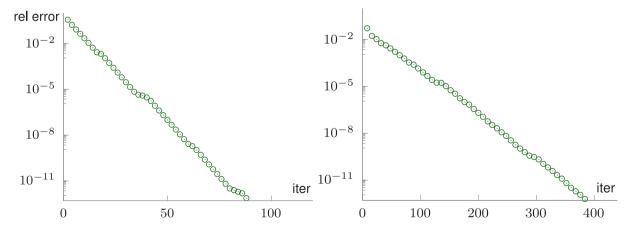


Fig. 3. An example of numerical convergence of the iterative method for (v_2, w_2) for the Stokes waves at $s_1 = 0.13660355$ (left) and $s_2 = 0.1407965471$ (right). The relative L^2 norm of the residual is shown versus the iteration number.

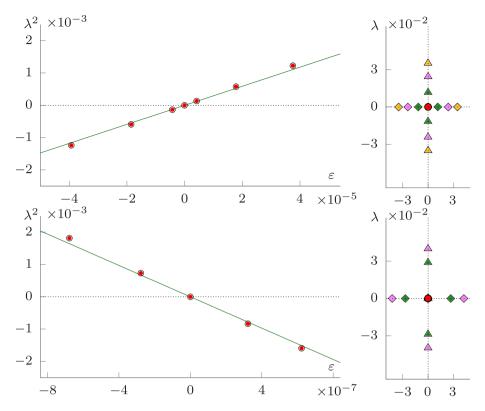


Fig. 4. Top left shows $\lambda^2(\varepsilon)$ obtained from numerical solution of the stability problem (18) (red dots), and evaluating the leading order (41) (green line) with $\lambda_1^2 = 29.4871$ ($\beta = 11.01822$); and top right shows a pair of real eigenvalues appearing from a collision of two imaginary eigenvalues for $c - c_0 = \epsilon = -3.93 \times 10^{-5}$, $\varepsilon = -1.85 \times 10^{-5}$, $\varepsilon = -4.23 \times 10^{-6}$ (gold, orchid and green triangles respectively), $\varepsilon = 0$ (red circle), $\varepsilon = 4.17 \times 10^{-6}$, $\varepsilon = 1.78 \times 10^{-5}$ and $\varepsilon = 3.75 \times 10^{-5}$ (green, orchid and gold diamonds respectively). Bottom row shows the associated quantities at the second extremum with $\lambda_1^2 = -2424.9$ (B = 10.96232) and $c - c_0 = \epsilon = 6.23 \times 10^{-7}$, $\varepsilon = 3.23 \times 10^{-7}$ (orchid and green triangles respectively), $\varepsilon = 0$ (red circle), $\varepsilon = -2.77 \times 10^{-7}$ and $\varepsilon = -6.77 \times 10^{-7}$ (green and orchid diamonds respectively).

CRediT authorship contribution statement

Sergey Dyachenko: Writing - review & editing, Writing - original draft, Visualization, Validation, Software, Investigation. Dmitry E. Pelinovsky: Writing - review & editing, Writing - original draft, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: the second author is Deputy Editor in Physica D. The first author declares no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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